PHYSICS 320: Problem Set No. 5
Due: Fri. Nov. 12 2010

1. The elastic free energy of a solid was shown to be of the form

$$E = E_0 + \frac{1}{2}\int d\mathbf{x}\frac{\partial r^\alpha}{\partial x^\mu}\bar{G}^{\mu\nu}_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\nu},$$

where the elastic tensor

$$\bar{G}^{\mu\nu}_{\alpha\beta} = -\frac{1}{2v}\sum_{\mathbf{R}} R^\mu D_{\alpha\beta}(\mathbf{R})R^\nu.$$

v is the volume of a unit cell and it was argued that for a Bravais lattice $D_{\alpha\beta}(\mathbf{R}) = D_{\beta\alpha}(\mathbf{R})$. The elastic free energy should also be unchanged under a rigid rotation of the lattice by any angle θ about any axis $\hat{\mathbf{n}}$.

   (a) Show that $\theta\hat{\mathbf{n}}$ is proportional to the antisymmetric part of the strain tensor $\frac{1}{2}\left(\frac{\partial r^\alpha}{\partial x^\beta} - \frac{\partial r^\beta}{\partial x^\alpha}\right)$ and hence that the elastic free energy can only depend on the symmetric strain tensor $\tilde{Y}^\alpha_\beta = \frac{1}{2}\left(\frac{\partial r^\alpha}{\partial x^\beta} + \frac{\partial r^\beta}{\partial x^\alpha}\right)$.

   (b) Thus,

$$E = E_0 + \frac{1}{2}\int d\mathbf{x}\tilde{Y}^\alpha_\mu\bar{C}^{\mu\nu}_{\alpha\beta}\tilde{Y}^\beta_\nu.$$

   What is $\bar{C}^{\mu\nu}_{\alpha\beta}$ in terms of $\bar{G}^{\mu\nu}_{\alpha\beta}$? How many independent components does it have in d dimensions?

2. Consider a one dimensional chain of ions interacting through pairwise potentials $f_{ij}(r)$. The ions form a lattice of spacing a.

   (a) If the interaction $f(r)$ is only between nearest neighbours, show that the Grüneisen parameters are independent of the momentum k and given by

$$\gamma = -\frac{a}{2}\frac{f'''(a)}{f''(a)}.$$

   (b) Now assume that there is also an interaction between next nearest neighbours. Show that the Grüneisen parameters will depend on k.

3. In the calculation of the renormalized phonon frequencies due to electron-phonon interactions, we encountered the expression

$$\hbar\omega^{(p)}_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - 2|M_{\mathbf{q}}|^2\sum_{\mathbf{k}}\frac{n_{\mathbf{k}}}{\epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{k}}}.$$

Evaluate $\hbar\omega^{(p)}_{\mathbf{q}} - \hbar\omega_{\mathbf{q}}$ by converting the sum into an integral in 1 and 3 dimensions and assuming that $\epsilon_{\mathbf{k}}$ is the free electron dispersion. Comment on the nature of the singularity at $|\mathbf{q}| = 2k_F$ in each case.

4. Consider an electron that moves through a deformable medium with bulk modulus B. Assume that the dynamics of the medium (but not the electron) is classical. You can think of a polaron in such a system as a bound state of the electron and a lattice distortion. The polaron will be stable if the binding energy of the electron is greater than the elastic energy of such a distortion. If the wavelength of the distortion is larger than a microscopic length scale (like the lattice spacing), the potential felt by the electron due to the distortion is $V(\mathbf{r}) = \alpha\nabla\cdot\mathbf{u}(\mathbf{r})$, where $\mathbf{u}(\mathbf{r})$ is the displacement field causing the distortion and α a constant.

   (a) Assume that the size of the distortion is L in d dimensions and that the potential felt by the electron is a square well potential. Determine the magnitude of $\nabla\cdot\mathbf{u}(\mathbf{r})$ that minimizes the energy of the system.

   (b) L can be thought of as the size of the polaron. Show that there is no polaron for $d > 2$. What is the size of the polaron for $d < 2$?

   (c) In reality, a polaron can exist in 3D since the lattice is discrete and the size of the bound state cannot be lower than the lattice spacing a. If we assume that the elastic energy is infinite for square well distortions with $L < a$, find the condition on α for a stable polaron to exist in 3D.