PHYSICS 320: Problem Set No. 3
Due: Wed. Oct. 4 2010

1. In class, we calculated the imaginary part of the dielectric constant $\epsilon_2(\mathbf{q},\omega)$ at $T = 0$ and marked out the region in the $(\omega, |\mathbf{q}|)$ plane where it is not equal to zero. We also argued that this region is where particle-hole excitations exist. In this problem you will see this directly.

   (a) Consider a free Fermi gas with Fermi momentum $\hbar k_F$ at $T = 0$. A particle-hole excitation is created when an electron from inside the Fermi sea is excited to a state outside it such that the momentum of the system has changed by an amount $\hbar\mathbf{q}$ and its energy by $\hbar\omega$. Indicate the region in the $(\omega, |\mathbf{q}|)$ plane where such particle-hole excitations exist.

   (b) How is the case of one dimension different from two and three dimensions?

2. Now consider a system at $T = 0$, where free electrons exist in two bands with dispersions

$$E_{\pm}(\mathbf{k}) = \pm\left(\frac{\hbar^2|\mathbf{k}|^2}{2m} + \frac{\Delta}{2}\right),$$

   The lower band is completely occupied and the electrons in the upper band have Fermi momentum $\hbar k_F$. Consider particle-hole excitations of the sort where electrons from the lower band are excited to the upper band and the energy and momentum of the system change by $\hbar\omega$ and $\hbar\mathbf{q}$. Indicate the region in the $(\omega, |\mathbf{q}|)$ plane, where such excitations exist. What role does dimensionality play here?

3. Consider the Kubo formula for the complex permittivity derived in class,

$$\tilde{\epsilon}(\mathbf{q},\omega) = \epsilon_1(\mathbf{q},\omega) + i\epsilon_2(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{q^2}\sum_{\mathbf{k}} \frac{f_0(\epsilon_{\mathbf{k}+\mathbf{q}}) - f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} + \omega + i\delta},$$

   with $\delta \to 0+$. Here $f_0(\epsilon_{\mathbf{k}})$ is the occupancy of the single particle state of a free fermion with momentum $\mathbf{k}$ and energy $\epsilon_{\mathbf{k}}$.

   (a) Obtain an expression for $\epsilon_2(\mathbf{q},\omega)]$ in the limit $\omega \gg (|\mathbf{q}|, k_F|\mathbf{q}|/m)$. You can leave the expression as a sum/integral over momenta. This is the Landau damping term responsible for damping out plasma oscillations at large wave number. You can see that only particles with specific values of $\mathbf{k}$ contribute to the damping of a particular $(\mathbf{q},\omega_{\mathbf{q}})$ plasma mode. What is the velocity of these particles in the direction of $\mathbf{q}$?

   (b) It was shown in class that plasma oscillations were longitudinal oscillations of the electric field. Transverse oscillations give rise to regular EM waves. What is the condition on $\epsilon_1(\mathbf{q},\omega)$ for EM waves to propagate? What is the dispersion relation for these waves at small $\mathbf{q}$?

   (c) At large values of $\mathbf{q}$, longitudinal plasma oscillations undergo Landau damping. What happens to transverse EM waves? (*Hint: Use the condition on the velocity of the particles contributing to Landau damping from part (a).*)

4. In class we saw that upon applying a magnetic field B the Fermi surface of a Fermi liquid deformed and the up (down) spin quasiparticles formed separate Fermi surfaces with Fermi momentum $k_{F\uparrow}$ $(k_{F\downarrow})$ such that

$$\tilde{\epsilon}_{k_{F\uparrow}\uparrow} + \frac{1}{2}g\mu_B B = \tilde{\epsilon}_{k_{F\downarrow}\downarrow} - \frac{1}{2}g\mu_B B,$$

   where g is the bare g factor of the electron. We can define an *effective g factor* (denoted by g^*) of the quasiparticle such that

$$\epsilon_{k_{F\uparrow}\uparrow} + \frac{1}{2}g^*\mu_B B = \epsilon_{k_{F\downarrow}\downarrow} - \frac{1}{2}g^*\mu_B B.$$

   To be consistent, the energy to flip the spin of a quasiparticle from $\downarrow$ to $\uparrow$ in the field B without changing anything else must also equal $g^*\mu_B B$.

   (a) Show that for non-interacting electrons, $g^* = g$, the bare g factor.

(b) Calculate g^* for a Fermi liquid in terms of g and $F_l^{(s,a)}$.

(c) Is it possible to flip the spin a quasiparticle from $\uparrow$ to $\downarrow$?

5. Calculate the Landau parameters $\epsilon_{\mathbf{k},\sigma}$, $f^s(\mathbf{k},\mathbf{k}')$ and $f^a(\mathbf{k},\mathbf{k}')$ in the Hartree-Fock approximation for the jellium model.

   (a) To calculate $\epsilon_{\mathbf{k},\sigma}$, consider the state of the system with a filled Fermi sphere of Fermi momentum $\hbar k_F$ and a single electron with momentum $\hbar\mathbf{k}$ and spin σ.

   (b) Now consider the case where you have a filled Fermi sphere and two electrons with momentum and spin values $(\hbar\mathbf{k},\sigma)$ and $(\hbar\mathbf{k}',\sigma')$. $f^s(\mathbf{k},\mathbf{k}')$ and $f^a(\mathbf{k},\mathbf{k}')$ are given by the interaction energy between the two electrons. Show using the equation for the effective mass in Fermi liquid theory that it goes to zero at the Fermi surface.