PHYSICS 320: Problem Set No. 0
Due: Wed. Aug. 18 2010

1. A particle of mass m moving in 1D in a potential $V(x)$.

   (a) Consider

   $$V(x) = -\frac{\hbar^2}{2m}\Delta\delta(x),$$

   with $\Delta > 0$. What is the energy of the lowest eigenstate?

   (b) Now, consider

   $$V(x) = -\frac{\hbar^2}{2m}\Delta\sum_n \delta(x - na),$$

   where n takes on all integer values and $\Delta > 0$. The energy eigenstates of this system are Bloch states that can be labelled with momentum k. Calculate the energy of the $k = 0$ state. You may leave the energy in terms of a transcendental equation.

   (c) Compare the energies obtained in (b) for $\Delta a = 5$ with the one obtained in (a). This might have to be done numerically.

2. Consider a system of N atoms of mass m constrained to move on a ring. Each atom is connected to its left and right neighbour by harmonic springs of spring constant Γ. In equilibrium each atom is a distance a away from its neighbours.

   (a) Using the classical equations of motion for the atoms obtain a formula for the eigenfrequencies of oscillation of this system.

   (b) Now, suppose one of the atoms in the chain is replaced by another of mass M. Obtain expressions for the eigenfrequencies of this system. You may leave the answer in the form of transcendental equations in frequency or wavenumber.

   (c) In the limit $N \to \infty$ with a fixed, obtain an expression for the maximum eigenfrequency as a function of M/m. What is the difference between the cases $M < m$ and $M > m$?

3. Evaluate the Fourier transform of the potential

   $$V(\mathbf{r}) = \frac{e^{-|\mathbf{r}|/\xi}}{|\mathbf{r}|}$$

   in two and three dimensions. (*Hint: In the two dimensional case, it might help to do the integral over r first and then θ in polar coordinates.*)

4. A particular level in a quantum system can accommodate at most two electrons, one with spin up and the other with spin down. The energy of the electrons in this level is given by

   $$E = \epsilon\,(n_\uparrow + n_\downarrow) + U n_\uparrow n_\downarrow,$$

   where $n_\uparrow$ and $n_\downarrow$ are the numbers of up and down electrons in it respectively. If the system is in equilibrium at temperature T and the chemical potential of the electrons is μ,

   (a) Calculate the average number and energy of the electrons in the level.

   (b) Evaluate the above two quantities in the limits, $U/|\epsilon - \mu| \to 0$, $U/|\epsilon - \mu| \to \infty$ and $U/|\epsilon - \mu| \to -\infty$.

5. The wavenumber and frequency dependent dielectric constant of a medium $\epsilon(\mathbf{k}, \omega)$ is given by

   $$\mathbf{D}(\mathbf{k}, \omega) = \epsilon(\mathbf{k}, \omega)\mathbf{E}(\mathbf{k}, \omega),$$

   where $\mathbf{E}(\mathbf{k}, \omega)$ is the Fourier transform of the (space and time dependent) electric field vector and $\mathbf{D}(\mathbf{k}, \omega)$, of the displacement vector.

(a) If there is no magnetic field, at what values of (ω, $\mathbf{k}$) will the system have self sustaining longitudinal electric field waves? Self sustaining here means a wave that can exist without free charges or currents driving it [*Hint: Use Maxwell's equations. The values of* ω *and* $\mathbf{k}$ *will emerge from a certain condition on* $\epsilon(\mathbf{k}, \omega)$.]. Can the system ever have longitudinal magnetic field waves?

(b) Consider a system of free electrons of density n obeying Newton's laws of motion. What is $\epsilon(\mathbf{k}, \omega)$ for this system? Show that it is independent of $\mathbf{k}$ and hence there is only frequency at which self sustaining longitudinal electric field oscillations can occur. What is this frequency equal to? [*Hint: Write down the equation of motion of an electron in the presence of an electric field wave. From this calculate the polarization vector of the system and hence* $\epsilon(\mathbf{k}, \omega)$.]

6. Consider a two dimensional square lattice of spacing a. Let z be the number of conduction electrons per unit cell.

(a) Draw the first three Brillouin zones in the extended zone scheme.

(b) Ignoring the periodic potential due to the lattice calculate the Fermi momentum k_F.

(c) Draw the Fermi surface for $z = 1, 2, 3$ and 4 in the extended and reduced zone schemes.