1 a) The volume V_d of a d-dimensional hypersphere is proportional to r^d (more precisely $V_d = \frac{\Omega_d r^d}{d}$, where Ω_d is the d-dimensional solid angle).

The density $n = \frac{1}{V_d}$, where V_d now is the volume of a sphere of radius $r_s a_0$. Thus

$$n \propto r_s^{-d} \Rightarrow r_s \propto n^{-1/d} \quad \text{or} \quad \alpha = -1/d$$

b) The ~~do~~ Fermi volume in d dimensions is proportional to p^d, where p is its radius. The density n is proportional to the Fermi volume of radius p_F.

Thus $p_F \propto n^{1/d}$.

$$E = \frac{1}{V_d} \sum_{\vec{p}} \frac{p^2}{2m} \propto \frac{1}{V_d} V_d \int_0^{p_F} p^{d+1} dp$$

$$\left(\because \sum_{\vec{p}} = V_d \int d^d \vec{p} \right) \propto p_F^{d+2} \propto n^{1+2/d}$$

But from ~~the previous~~ part a), $n \propto r_s^{-d}$

Thus $E \propto r_s^{-(d+2)}$ and $\beta = d+2$

c) $\varepsilon_{ex}(\vec{p}) = -\frac{e^2}{V_d} \sum_{\vec{p}'} \int d^d\vec{r}\, \frac{e^{i(\vec{p}-\vec{p}')\cdot\vec{r}/\hbar}}{|\vec{r}|}$

$\propto -\frac{1}{V_d} \sum_{\vec{p}'} \int d\Omega_d \int_0^\infty dr\, r^{d-1}\, \frac{e^{i|\vec{p}-\vec{p}'|r\cos\theta}}{r}$

where θ is the angle between $\vec{p}-\vec{p}'$ and $\vec{r}$. $d\Omega_d$ is the solid angle element in d dimensions.

We can pull out factors of $|\vec{p}-\vec{p}'|$ from the above integral in such a way that the resultant integral is just a number.

$$\int d\Omega_d \int_0^\infty dr\, r^{d-1}\, \frac{e^{i|\vec{p}-\vec{p}'|r\cos\theta}}{r} = \frac{1}{|\vec{p}-\vec{p}'|^{d-1}} I,$$

where $I = \int d\Omega_d \int_0^\infty du\, u^{d-2} e^{iu\cos\theta}$, which is just a number.

Thus $\varepsilon_{ex}(\vec{p}) \propto -\frac{1}{V_d} \sum_{\vec{p}'} \frac{1}{|\vec{p}-\vec{p}'|^{d-1}}$.

and $E_{ex} = \frac{1}{2V_d} \sum_{\vec{p}} \varepsilon_{ex}(\vec{p}) \propto -\frac{1}{V_d^2} \sum_{\vec{p},\vec{p}'} \frac{1}{|\vec{p}-\vec{p}'|^{d-1}}$

$\propto -\frac{1}{V_d^2} V_d^2 \int d\Omega_d \int d\Omega_d' \int_0^{p_F} dp \int_0^{p_F} dp'\, \frac{p^{d-1}(p')^{d-1}}{|\vec{p}-\vec{p}'|^{d-1}}$

where we have once again used the fact that $\sum_{\vec{p}} = V_d \int d^d\vec{p}$.

Note that we cannot perform the angular integrations over $d\Omega_d$ and $d\Omega'_d$ to get two factors of 4π since the $\frac{1}{|\vec{p}-\vec{p}'|^{d-1}}$ factor in the integrand depends on the angular variables. We can now pull out factors of p_F from the integral to obtain

$E_{ex} \propto -p_F^{d+1} I$, where

$$I = \int d\Omega_d \int d\Omega'_d \int_0^1 du \int_0^1 du' \frac{u^{d-1}(u')^{d-1}}{|\vec{u}-\vec{u}'|^{d-1}}$$, which

is just a number.

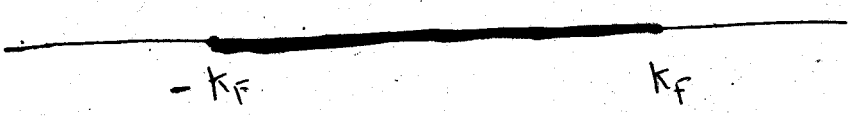
Thus $E_{ex} \propto -p_F^{d+1} \propto -r_s^{-(d+1)}$ from parts a) and b)

$\Rightarrow \gamma = d+1$

Parts b) and c) can also be done in a simpler way. The KE per particle has to be $\propto p_F^2$. But from part b), $p_F \propto r_s^{-1}$. Thus the KE per particle $\propto r_s^{-2}$. The # of particles per unit volume $\propto r_s^{-d}$ (from part a). Thus, the KE per unit volume $\propto r_s^{-(d+2)}$. giving $\beta = d+2$.

Similarly the exchange energy per particle has to be $\propto \frac{1}{r_s}$ (the form of the interaction is $1/r$ and r_s is the only length scale in the problem.). The exchange energy per unit volume is thus $\frac{1}{r_s^{d+1}}$, giving $\gamma = d+1$.

2 a) The Fermi surface is a line segment going from $-k_F$ to k_F.



Consider a momentum value $q_f > 0$. If an excitation of momentum q_f is created by exciting a fermion from a single particle state of momentum k, the energy of the excited state is $\frac{\hbar^2 (k+q_f)^2}{2m}$. The excitation energy $\Delta E_{q_f} = \frac{\hbar^2}{2m}\left[(k+q_f)^2 - k^2\right] = \frac{\hbar^2}{2m}\left(q_f^2 + 2kq_f\right)$

Since ΔE_{q_f} is a monotonically increasing function of k for $q_f > 0$, ΔE_{q_f} is largest for the largest possible value of $k = k_F$.

Thus $\Delta E_q^{max} = \frac{\hbar^2}{2m} q (q + 2k_F)$.

The smallest possible value of ΔE_q is when k takes on the smallest possible value between ($-k_F$ and k_F) and $k+q$ is outside the Fermi surface. For $q < 2k_F$, this happens when $k+q = k_F$ or $k = k_F - q$, the corresponding $\Delta E_q^{min} = \frac{\hbar^2}{2m} q (2k_F - q)$.

For $q > 2k_F$, $k = k_F - q < -k_F$ and is not allowed.

Thus k remains fixed at $k = -k_F$. $\Rightarrow \Delta E_q^{min} = \frac{\hbar^2}{2m} q (q - 2k_F)$

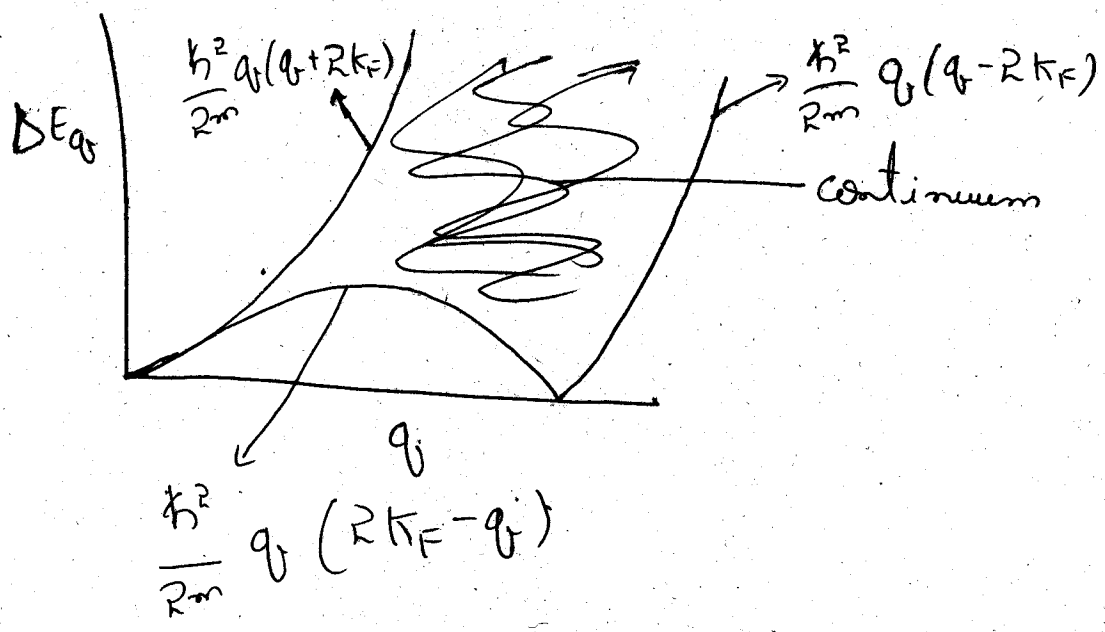
So, we finally obtain

$$\Delta E_q^{max} = \frac{\hbar^2}{2m} q (q + 2k_F) \quad \forall q \geq 0$$

$$\Delta E_q^{min} = \frac{\hbar^2}{2m} q (2k_F - q) \quad \forall 0 \leq q \leq 2k_F$$

$$= \frac{\hbar^2}{2m} q (q - 2k_F) \quad \forall q \geq 2k_F$$

The plot of the particle-hole continuum is thus



The plot is symmetric about $q_f=0$ and its mirror image about the ΔE_q axis is what you obtain for $q_f<0$.

b) For $q_f \to 0$ $\quad \frac{\Delta E_{q_f}^{max} + \Delta E_{q_f}^{min}}{2} = \frac{\hbar^2}{2m} 2k_F q_f = E_{av}(\vec{q}_f)$

and $\quad \Delta E_{q_f}^{max} - \Delta E_{q_f}^{min} = \frac{\hbar^2}{2m} 2q_f^2 = \Delta(\vec{q}_f)$

Thus $\lim_{\vec{q}_f \to 0} \frac{\Delta(\vec{q}_f)}{E_{av}(\vec{q}_f)} = \frac{q_f}{k_F} \to 0$

3a) The Hamiltonian

$$H = -t \sum_j c_j^\dagger c_{j+1} + h.c. + \varepsilon_1 \sum_{i \in odd} n_i + \varepsilon_2 \sum_{i \in even} n_i$$

$c_i^\dagger$ creates a down spin electron at site i and $n_i = c_i^\dagger c_i$.

The pattern of alternate immobile up spin electrons creates a staggered array of on site energies for the down spin electrons. Thus, there are two on-site energies, one for the odd numbered sites and one for the even numbered ones. If the up spin electrons are at even numbered sites, $\varepsilon_1 = 0$ and $\varepsilon_2 = U$.

Thus, the Hamiltonian is

$$H = -t \sum_j c_j^\dagger c_{j+1} + h.c. + \varepsilon_1 \sum_{i \text{ odd}} n_i + \varepsilon_2 \sum_{i \in \text{even}} n_i$$

$$\varepsilon_1 = 0 \; ; \; \varepsilon_2 = U$$

b) Let us define a and b operators such that

$$a_i^\dagger = c_i^\dagger \quad \text{for } i \in \text{odd}$$
$$b_i^\dagger = c_{i+1}^\dagger$$

Thus $a_i^\dagger$ creates a down spin electron at the odd numbered site i and $b_i^\dagger$ at the even numbered site $i+1$. ~~The~~

~~Ham~~

The Hamiltonian

$$H = -t \sum_j a_j^\dagger b_j + b_j^\dagger a_{j+1} + h.c. + \varepsilon_1 \sum_j n_{aj} + \varepsilon_2 \sum_j n_{bj}$$

where j goes only over the odd numbered sites. The sites j form a lattice of spacing $2a$ (the unit cell has now been doubled)

$$a_j^\dagger = \frac{1}{\sqrt{m}} \sum_k e^{ikj2a} a_k^\dagger$$

$$b_j^\dagger = \frac{1}{\sqrt{m}} \sum_k e^{ikj2a} b_k^\dagger$$

where j once again goes only over the odd sites. $m = N/2$, where N is the total # of sites. k runs from $-\frac{\pi}{2a}$ to $\frac{\pi}{2a}$.

$$H = -\frac{t}{m} \sum_{jk_1k_2} \left[a_{k_1}^\dagger b_{k_2} e^{i(k_1-k_2)j2a} + b_{k_1}^\dagger a_{k_2} e^{i(k_1-k_2)j2a} \right.$$
$$\left. + b_{k_1}^\dagger a_{k_2} e^{i(k_1-k_2)j2a} e^{-ik_2 2a} + a_{k_1}^\dagger b_{k_2} e^{i(k_1-k_2)j2a} e^{-ik_1 2a} \right]$$
$$+ \varepsilon_1 \sum_k a_k^\dagger a_k + \varepsilon_2 \sum_k b_k^\dagger b_k$$

$$= \sum_k \begin{pmatrix} a_k^\dagger & b_k^\dagger \end{pmatrix} \begin{pmatrix} \varepsilon_1 & -t\left[1+e^{ikR_2a}\right] \\ -t\left[1+e^{-ikR_2a}\right] & \varepsilon_2 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

Since $\sum_{q_j} e^{iq_jR_2a} = \delta(q_j)$

The matrix $\begin{pmatrix} \varepsilon_1 & -t\left[1+e^{ikR_2a}\right] \\ -t\left[1+e^{-ikR_2a}\right] & \varepsilon_2 \end{pmatrix}$ has eigenvalues given by

$$(\varepsilon_k - \varepsilon_1)(\varepsilon_k - \varepsilon_2) - t^2(2 + 2\cos 2ka) = 0$$

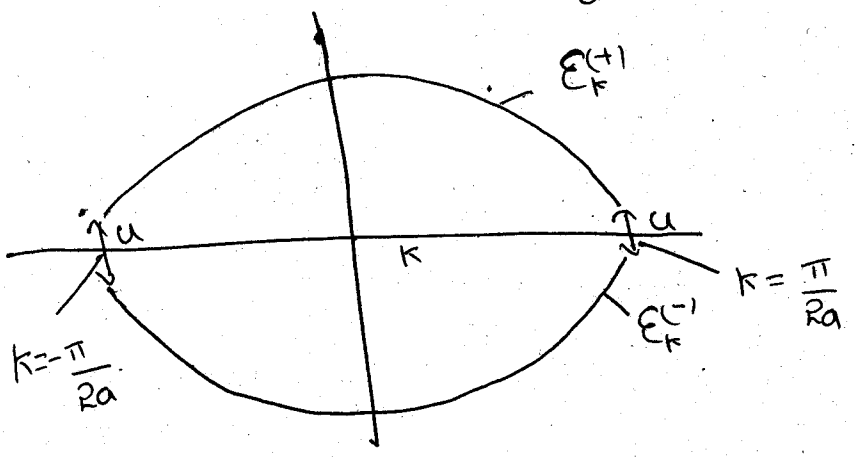
$$\Rightarrow \varepsilon_k = \frac{\varepsilon_1 + \varepsilon_2 \pm \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 8t^2(1+\cos 2ka)}}{2}$$

Putting in $\varepsilon_1 = 0$ and $\varepsilon_2 = u$, we obtain the two bands.

$$\varepsilon_k^{(+)} = \frac{u + \sqrt{u^2 + 8t^2(1+\cos 2ka)}}{2}$$

and $$\varepsilon_k^{(-)} = \frac{u - \sqrt{u^2 + 8t^2(1+\cos 2ka)}}{2}$$

These can be plotted to give



This problem is a lot like problem 2 from problem set #2. In that problem too one obtains two bands with a gap from the doubling of the unit cell. The doubling there comes from having ~~two~~ different values of the hopping on even and odd bonds. In this problem the doubling comes from having different on-site energies on even and odd sites. Just like in the other problem, where we obtain only one band in the limit $t_1 \to t_2$, here too we obtain only one band as $U \to 0$.