PHYSICS 320: Problem Set No. 5
Due: Mon. Oct. 26 2009

Only problems 1-4 need to be turned in for credit. Problem 5 is optional.

1. Consider a two-site Hubbard model with the Hamiltonian
$$H = -t \sum_{\sigma} c^{\dagger}_{1\sigma} c_{2\sigma} + c^{\dagger}_{2\sigma} c_{1\sigma} + U \left[n_{1\uparrow} n_{1\downarrow} + n_{2\uparrow} n_{2\downarrow} \right].$$
If there are two electrons and $|t/U| \ll 1$, show that second order perturbation theory generates a Hamiltonian of the form $H = J\mathbf{S}_1.\mathbf{S}_2$ between the spins of the electrons on the two sites. What is the value of J?

2. Consider the Hubbard model
$$H = U \sum_i n_{i\uparrow} n_{i\downarrow}$$
without the hopping term and $U > 0$.
   (a) At $T = 0$, the chemical potential is the energy required to add a particle to the system, i.e. $\mu = E(N+1) - E(N)$, where $E(N)$ is the energy of the system with N particles. Calculate the chemical potential as a function of the filling n.
   (b) Now consider $T \neq 0$. Once again calculate μ as a function of n. It will help to use the partition function in the grand canonical ensemble. Show that you obtain the same value of μ as part (a) in the limit $T \to 0$.

3. In this problem, you will calculate the energy spectrum and degeneracy of an electron in a magnetic field in a gauge invariant form (i. e. you will not choose a value for $\nabla.\mathbf{A}$ where $\mathbf{A}$ is the magnetic vector potential). The field B is constant and in the $\hat{z}$ direction and the electron's motion is confined to the $x-y$ plane. The Hamiltonian is of the form $H = \frac{1}{2m_e}\left(\Pi_x^2 + \Pi_y^2\right)$, where $\Pi_\alpha = p_\alpha + \frac{e}{c}A_\alpha$.
   (a) Show that $[\Pi_x, \Pi_y] = iC$, is a gauge invariant quantity and C is a constant that depends on B and fundamental constants. Use this fact to show that the allowed energy eigenvalues have the form $E_n = (n+1/2)\hbar\omega$, where ω is the cyclotron frequency. You need not carry out the full calculation. Just use an analogy with a simple one dimensional quantum mechanical problem.
   (b) Show that you can construct operators X_0 and Y_0 with dimensions of length as linear combinations of the operators in the set (x, y, Π_x, Π_y) such that $[X_0, Y_0] = il^2$ (l is the magnetic length) and $[X_0, H] = [Y_0, H] = 0$.
   (c) Use the commutation relations derived in part (b) to argue that there is a degenerate set of states at each energy E_n that can be labelled by an integer m. (*Hint: Define raising and lowering operators* $a^\dagger$ *and* a *that commute with* H.)
   (d) Calculate the degeneracy of each energy level for a system of area L_xL_y in the following way: Since $[X_0, Y_0] = il^2$, X_0 and Y_0 can be thought of as canonically conjugate variable in a semi-classical approximation. Compute the number of states within a phase space area $\Delta X_0 \Delta Y_0$. The ranges of X_0 and Y_0 are L_x and L_y respectively. Show that this gives you the same degeneracy for each Landau level as obtained in class by making a gauge choice.

4. In class we used the method of adiabatic pumping to show that a single electron is transferred from one edge to another upon threading a quantum of flux when *all* Landau levels are filled in a clean sample (the Laughlin gauge argument). This gives a Hall resistance $R_H = -h/e^2 = -B/nec$, where B is the field and n, the density of electrons. Apply the method of adiabatic pumping to a *partially filled* Landau level to show that in this case too $R_H = -B/nec$. Thus, argue that quantum mechanically too one obtains the classical result $R_H = -B/nec$ for any filling n for a clean sample.

5. The Laughlin gauge argument relies on the fact that upon threading a flux quantum, the Hamiltonian of a system remains unchanged. Thus, energy levels that move can only take up the positions of other energy levels.

This conclusion applies more generally than to a system of non-interacting electrons in a magnetic field. Show that a general Hamiltonian of N particles of the form

$$H = \frac{1}{2m} \sum_{i=1}^{N} \left(\mathbf{p}_i - \frac{e}{c} \mathbf{A}_i \right)^2 + V\left(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n\right),$$

with periodic boundary conditions in the x direction remains invariant upon threading a flux quantum. Here $V\left(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n\right)$ represents any combination of one, two, $\ldots N$ body interactions. A flux quantum can be threaded by making the transformation $\mathbf{A}_i \rightarrow \mathbf{A}_i + \frac{\Phi_0}{L_x}\hat{x}$. $\Phi_0 = hc/e$ and L_x is the length in the x direction. It might help to consider H in second quantized form.