PHYSICS 320: Problem Set No. 4
Due: Wed. Sep. 30 2009

Only problems 1-4 need to be turned in for credit. The other two problems are optional. $\Re e(z)$ *and* $\Im m(z)$ *stand for the real and imaginary parts of* z *respectively.*

1. Consider a free Fermi gas with Fermi momentum k_F and corresponding Fermi energy E_F. The ground state is a Fermi sea with single particle states of energy $\epsilon_{\mathbf{k}} \leq E_F$ occupied. An excitation about the ground state is produced by moving a single fermion from inside the Fermi sea to an unoccupied state outside. This leaves a hole in the state from which the electron was moved and the excitation is thus called a particle-hole excitation. The momentum of the system changes by $\mathbf{q}$ in the process and its energy by $\hbar\omega$.

   (a) For what values of the combination $(\omega, |\mathbf{q}|)$ do such excitations exist? Make a sketch on a plot of $|\mathbf{q}|/2k_F$ and $\hbar\omega/E_F$. Show that it agrees with the plot we obtained in class from the condition $\Im m[\tilde{\epsilon}(\mathbf{q},\omega)] = 0$.

   (b) How is the case of one dimension qualitatively different from two and three dimensions?

2. Consider the Kubo formula for the complex permittivity derived in class,

$$\tilde{\epsilon}(\mathbf{q},\omega) = 1 - \frac{4\pi e^2}{q^2} \sum_{\mathbf{k}} \frac{f_0(\epsilon_{\mathbf{k}+\mathbf{q}}) - f_0(\epsilon_{\mathbf{k}})}{\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} + \omega + i\delta},$$

   with $\delta \to 0+$. Here $f_0(\epsilon_{\mathbf{k}})$ is the occupancy of the single particle state of a free fermion with momentum $\mathbf{k}$ and energy $\epsilon_{\mathbf{k}}$.

   (a) Obtain an expression for $\Im m[\tilde{\epsilon}(\mathbf{q},\omega)]$ in the limit $\omega \gg (|\mathbf{q}|, k_F|\mathbf{q}|/m)$. You can leave the expression as a sum/integral over momenta. This is the Landau damping term responsible for damping out plasma oscillations at large wave number. You can see that only particles with specific values of $\mathbf{k}$ contribute to the damping of a particular $(\mathbf{q}, \omega_{\mathbf{q}})$ plasma mode. What is the velocity of these particles in the direction of $\mathbf{q}$?

   (b) It was shown in class that plasma oscillations were longitudinal oscillations of the electric field. Transverse oscillations give rise to regular EM waves. What is the condition on $\Re e[\tilde{\epsilon}(\mathbf{q},\omega)]$ for EM waves to propagate? What is the dispersion relation for these waves at small $\mathbf{q}$?

   (c) At large values of $\mathbf{q}$, longitudinal plasma oscillations undergo Landau damping. What happens to transverse EM waves? (*Hint: Use the condition on the velocity of the particles contributing to Landau damping from part (a).*)

3. In class, we calculated the paramagnetic spin susceptibility of a Fermi liquid. This involved applying a magnetic field and creating an unequal population of up and down spin quasiparticles resulting in a non-zero magnetization. Now assume that in this configuration, we flip the spin of a quasiparticle from down (lower Zeeman energy) to up (higher Zeeman energy). This will change the magnetization of the system but if we demand that the magnetization does not change, there will have to be a transfer of quasiparticles from the up state to the down state. This will cause a net increase $\delta N_\downarrow$ in the number of down spin quasiparticles. The effective g factor g^* can be defined in this way. It is given by the equation $g^*\mu B = \Delta E$, where ΔE is the difference in energy of the state before and after the flip holding magnetization constant.

   (a) What is the relation between ΔE and $\delta N_\downarrow$? This expression will involve the bare Zeeman energy and the Fermi liquid parameters $f_l^{s,a}$.

   (b) Identifying g^* as the effective g factor also means that $\delta N_\downarrow = \frac{N^*(0)}{2}\frac{g^*\mu B}{2}$. What is the value of g^*/g in terms of $F_l^{s,a}$? Write down the expression for χ/χ_0 in terms of m^*/m and g^*/g where $\chi(\chi_0)$ is the paramaganetic susceptibility for the Fermi liquid (free Fermi gas).

4. The scattering rate of a electron out of a state labelled by momentum $\mathbf{k}$ and spin σ slightly above the Fermi surface is given by the following expression according to the Fermi golden rule:

$$\frac{1}{\tau_{\mathbf{k}\sigma}} = \frac{2\pi}{\hbar} \sum_{\mathbf{k}',\sigma',\mathbf{q}} |V(\mathbf{q})|^2 n_{\mathbf{k}'\sigma'} \left(1 - n_{\mathbf{k}'+\mathbf{q}\sigma'}\right)\left(1 - n_{\mathbf{k}-\mathbf{q}\sigma}\right) \delta\left(\epsilon_{\mathbf{k}'+\mathbf{q}\sigma'} + \epsilon_{\mathbf{k}-\mathbf{q}\sigma} - \epsilon_{\mathbf{k}'\sigma'} - \epsilon_{\mathbf{k}\sigma}\right).$$

Physically, this corresponds to the electron in state $(\mathbf{k}, \sigma)$ interacting with one in state $(\mathbf{k}', \sigma')$ and the pair getting scattered into states $(\mathbf{k} - \mathbf{q}, \sigma)$ and $(\mathbf{k}' + \mathbf{q}, \sigma')$ conserving energy and momentum. $\epsilon_{\mathbf{k}\sigma}$ is the energy of the single particle state labelled by $\mathbf{k}$ and spin σ and $n_{\mathbf{k}\sigma}$ its occupancy. $V(\mathbf{q})$ is the Fourier component of the two body potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$. Calculate the scattering rate in the following way.

(a) Show that

$$\frac{1}{\tau_{\mathbf{k}\sigma}} = \sum_{\mathbf{q}} \frac{q^2}{2\hbar e^2} |V(\mathbf{q})|^2 \left(1 - n_{\mathbf{k}-\mathbf{q}\sigma}\right) \frac{\Im m[\tilde{\epsilon}(\mathbf{q}, \omega_{\mathbf{kq}\sigma})]}{1 - e^{\beta \omega_{\mathbf{kq}\sigma}}},$$

where $\omega_{\mathbf{kq}\sigma} = \epsilon_{\mathbf{k}-\mathbf{q}\sigma} - \epsilon_{\mathbf{k}\sigma}$. Remember that the occupancies $\{n_{\mathbf{k}\sigma}\}$ obey the Fermi-Dirac distribution at temperature $T = 1/k_B\beta$.

(b) Take the limit $T \to 0$. Convert the sum into an integral over $\mathbf{q}$. What is the volume over which the integrand is non-zero? Since $|\mathbf{k}| \approx k_F$, this volume is small and consequently so are the values of $|\mathbf{q}|$. Use the expressions for $V(\mathbf{q})$ from Thomas-Fermi theory and $\Im m[\tilde{\epsilon}(\mathbf{q}, \omega_{\mathbf{kq}\sigma})]$ in the RPA in this limit to calculate $1/\tau_{\mathbf{k}\sigma}$.

5. (*Not for credit*) Consider again the expression for the complex permittivity given in problem #2,

(a) Calculate $\tilde{\epsilon}(\mathbf{q}, \omega)$ in the limit $\omega \to 0$ first and then $\mathbf{q} \to 0$. Show that in this limit, the permittivity agrees with the result from the linearized version of the Thomas-Fermi approximation.

(b) Now, consider the opposite limit $\mathbf{q} \to 0$ and then $\omega \to 0$ and calculate $\tilde{\epsilon}(\mathbf{q}, \omega)$. Show that the expression for the complex conductivity $\tilde{\sigma}(\mathbf{q}, \omega) = i\omega(\tilde{\epsilon}(\mathbf{q}, \omega) - 1)$ in this limit agrees with the one from ordinary Drude theory with the relaxation time $\tau \to 0$.

6. (*Not for credit*) Calculate the Landau parameters $\epsilon_{\mathbf{k},\sigma}$, $f^s(\mathbf{k}, \mathbf{k}')$ and $f^a(\mathbf{k}, \mathbf{k}')$ in the Hartree-Fock approximation for the jellium model.

(a) To calculate $\epsilon_{\mathbf{k},\sigma}$, consider the state of the system with a filled Fermi sphere of Fermi momentum $\hbar k_F$ and a single electron with momentum $\hbar\mathbf{k}$ and spin σ.

(b) Now consider the case where you have a filled Fermi sphere and two electrons with momentum and spin values $(\hbar\mathbf{k}, \sigma)$ and $(\hbar\mathbf{k}', \sigma')$. $f^s(\mathbf{k}, \mathbf{k}')$ and $f^a(\mathbf{k}, \mathbf{k}')$ are given by the interaction energy between the two electrons. Show using the equation for the effective mass in Fermi liquid theory that it goes to zero at the Fermi surface.