PH 206: Electromagnetic theory

Problem Set 9

1. Consider a rectangular wave guide of side lengths a and b. Obtain all the components of the electric and magnetic fields in the transverse magnetic $\mathrm{TM}_{m,n}$ mode. What is the cutoff frequency for each mode?

2. Calculate the Green's function for the d'Alembertian operator $\Box^2$. Remember that the Green's function is defined as

$$\Box^2 G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

The simplest way to proceed is to use Fourier transforms. Show that one can obtain both the retarded and advanced Green's functions in this way.

3. In class we argued that the equation

$$|\mathbf{r}-\mathbf{r}_0(t')| = c(t-t'),$$

arising in the calculation of the Liénard-Wiechert potentials of a point charge q cannot have more than one solution for the retarded time t' in terms of $\mathbf{r}$ and t for a given trajectory $\mathbf{r}_0(t)$. However, it is possible that there are no solutions for a given trajectory at certain values of $\mathbf{r}$ and t. An example of such a trajectory is

$$\mathbf{r}_0(t) = \sqrt{a^2 + c^2t^2}\,\hat{z},$$

with $-\infty < t < \infty$.

   (a) Consider $\mathbf{r} = z\hat{z}$. Show that for every value of z, there is a value of time $\tau(z)$ such that there are no solutions for $t < \tau(z)$. Calculate $\tau(z)$. Remember, we are only interested in solutions for t' which obey $t' < t$ (otherwise causality would be violated). This problem does not require complicated algebra. Draw the trajectory of the particle on a graph of z vs. ct. From each point on the trajectory draw light rays that move *forward* in time but in either direction in space. The points of intersection of such lines with a line at fixed z (which is where the potentials are to be calculated) will give you the function $\tau(z)$.

   (b) The potentials at $\mathbf{r} = z\hat{z}$ are zero for $t < \tau(z)$ since no "signal" from the point charge has reached. Calculate the potentials ϕ and $\mathbf{A}$ for $t > \tau(z)$.

4. In this problem, you will calculate the total energy radiated by a particle of charge q and mass m released with a speed v from infinity in the direction of a point charge Q held fixed so that the force on it is repulsive. Assume that the speed of v is non-relativistic throughout and you can ignore the energy lost due to radiation in comparison to the initial kinetic energy of the particle.

   (a) First assume that the charge q moves in the direction of Q, i.e. the impact parameter of the collision is zero. What is the total energy radiated?

   (b) Now, suppose that the charged particle q has impact parameter b. What is the total energy radiated? Show that in the limit $b \to 0$, you recover the expression of the previous part.

5. Consider a situation where the source charge and current densities oscillate with a period ω in time as

$$\begin{aligned}\mathbf{J}(\mathbf{r},t) &= \mathbf{J}(\mathbf{r})e^{-i\omega t}\\ \rho(\mathbf{r},t) &= \rho(\mathbf{r})e^{-i\omega t}.\end{aligned}$$

From the form of Maxwell's equations expressed in terms of the potentials, $\mathbf{A}(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$ it is clear that the potential also have the same time dependence, i.e.

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \mathbf{A}(\mathbf{r})e^{-i\omega t}\\ \phi(\mathbf{r},t) &= \phi(\mathbf{r})e^{-i\omega t}.\end{aligned}$$

Assume that the source has a typical size d and you want to calculate the electric field at point $\mathbf{r}$, such that $|\mathbf{r}-\mathbf{r}'| \gg \lambda \gg d$ for all points r' on the source, where $\lambda = 2\pi c/\omega$. The magnetic field and electric field will also have the time dependence $e^{-i\omega t}$ and from Maxwell's equations, they have the following spatial dependence:

$$\mathbf{B}(\mathbf{r}) = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})$$
$$\mathbf{E}(\mathbf{r}) = i\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{k}\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r}).$$

(a) Show from the form of the Liénard-Wiechert potentials that $\mathbf{A}(\mathbf{r})$ has the following form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_n \frac{(-ik)^n}{n!}\mathbf{M}_n,$$

where $\mathbf{M}_n$ is an appropriate n^{th} moment of the source distribution.

(b) What is the contribution from the $n = 0$ term above to $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$. Show that they are proportional to the electric dipole moment of the source.

(c) Now, calculate the contribution to to $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ from the $n = 1$ term. Show that there are two parts, one which is proportional to the magnetic dipole moment and the other to the electric quadrupole moment.