PH 206: Electromagnetic theory

Problem Set 8

1. A beam of light of electric field amplitude E_0 is incident from medium 1 onto the interface separating it from medium 2 at an angle of incidence θ. The incident beam is linearly polarized perpendicular to the plane of incidence. Calculate the amplitude of the reflected and transmitted beams of light. Is there a Brewester angle?

2. An anisotropic linear dielectric obeys the following relation between the $\mathbf{D}$ and $\mathbf{E}$ vectors,

$$D_\alpha(\mathbf{r}, t) = \epsilon_{\alpha\beta} E_\alpha(\mathbf{r}, t).$$

   Here α and β take on the values x, y and z and the permittivity $\epsilon_{\alpha\beta}$ is thus a second rank tensor. Further, let $\epsilon_{xx} = \epsilon_{yy} = \epsilon_1$, $\epsilon_{zz} = \epsilon_2$ and $\epsilon_{xy} = \epsilon_{yx} = \epsilon_{yz} = \epsilon_{zy} = \epsilon_{zx} = \epsilon_{xz} = 0$. The permeability is μ_0.

   (a) Work out the dispersion relation between the wavevector $\mathbf{k}$ and the frequency ω. Show that there are two distinct dispersion relations.

   (b) Suppose a light wave propagates with a wavevector $k\hat{x}$. What are the two possible frequencies and what is the polarization associated with each?

   (c) Repeat the above exercise for a wave with wavevector $k\hat{z}$.

   (d) Suppose light of frequency ω travelling in the $\hat{x}$ direction in the dielectric has an electric field vector that oscillates in the direction $\cos\theta\hat{y} + \sin\theta\hat{z}$ at $x = 0$. What is the direction of the electric field vector's oscillations at $x = a$?

3. The most general linear relation between the $\mathbf{D}$ and $\mathbf{E}$ vectors of an isotropic linear dielectric is

$$\mathbf{D}(\mathbf{r}, t) = \int d\mathbf{r}' \int dt' \epsilon(\mathbf{r} - \mathbf{r}, t - t')\mathbf{E}(\mathbf{r}', t').$$

   The magnetic permeability of the dielectric is μ_0.

   (a) Write down Maxwell's equation in Fourier space, i.e. write them down in terms of $\mathbf{E}(\mathbf{k}, \omega)$, $\mathbf{B}(\mathbf{k}, \omega)$, $\epsilon(\mathbf{k}, \omega)$, $\rho_f(\mathbf{k}, \omega)$ and $\mathbf{J}_f(\mathbf{k}, \omega)$, which are the Fourier transforms of the electric field, magnetic field, permittivity, free charge density and free current density.

   (b) If self-sustaining transverse electric and magnetic fields (i.e. transverse fields which are non-zero even in the absence of free charges and currents) exist in the dielectric at wavevector $\mathbf{k}$ and frequency ω, what is the equation that relates $\mathbf{k}$ and ω?

   (c) If self-sustaining longitudinal electric field waves exist at wavevector $\mathbf{k}$ and ω, what is the relation between $\mathbf{k}$ and ω? Such waves are called longitudinal plasma waves. What is the magnetic field in such a wave?

   (d) Can longitudinal magnetic field waves be set up? If so, how? If not, why?

4. Let $|x\rangle$ denote a plane wave of light of wave vector $k\hat{z}$ and frequency ω of unit amplitude linearly polarized along $\hat{x}$. Similarly let $|y\rangle$ denote a plane wave of light of wave vector $k\hat{z}$ and frequency ω of unit amplitude linearly polarized along $\hat{y}$. The general state of polarization $|p\rangle$ of a plane wave with wave vector $k\hat{z}$ and frequency ω is a linear superposition of $|x\rangle$ and $|y\rangle$. One can then make an a analogy with a spin 1/2 system in quantum mechanics with $|x\rangle$ and $|y\rangle$ analogous to $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively.

   (a) How many real numbers are required to specify the state of polarization in this representation? What about for the general state of a spin 1/2 system? Why is there a difference?

   (b) One can also represent unpolarized and partially polarized light using this convention and the concept of density matrices like in quantum mechanics. Light which is not fully polarized can be thought of as being an incoherent combination of fully polarized light beams $|p_i\rangle$ with probabilities w_i and can be represented by the density matrix

$$\rho = \sum_i w_i |p_i\rangle\langle p_i|.$$

i. What is the intensity of the light described by ρ?

ii. How many real numbers are required to specify ρ? How many are required for the density matrix of a spin 1/2 system? Why is there a difference?

iii. What are the eigenvalues of ρ for completely polarized light of intensity I?

iv. What is the form of ρ for completely unpolarized light of intensity I? Why?

5. Consider a beam of polarized light incident from medium 1 at angle θ_i from the normal onto an interface separating it from medium 2. The amplitude of the incident beam is E and the amplitudes of the reflected and transmitted beams are rE and tE, where r and t are the reflection and transmission coefficients respectively. The transmitted beam makes an angle θ_t with the normal according to Snell's law. Now, consider an incident beam from medium 2 at angle of incidence θ_t from the normal onto the the interface. You can think of this beam as moving in a direction exactly opposite to the transmitted beam. The coefficients of reflection and transmission for this new beam are r' and t' respectively. Further, the new reflected beam is in medium 2 and the new transmitted beam is in medium 1 making an angle θ_i with the normal in accordance with Snell's law.

(a) Argue that $r' = -r$ and $t' = t$. (*Hint: Use the fact that if in the first situation you reversed the direction of the reflected and transmitted beams, they would combine to produce a beam with the same amplitude as the incident beam but moving in the opposite direction.*)

(b) In general the following matrix relation exists between the amplitudes of the different beams

$$\begin{pmatrix} E_2^{(+)} \\ E_2^{(-)} \end{pmatrix} = M \begin{pmatrix} E_1^{(+)} \\ E_1^{(-)} \end{pmatrix},$$

where 1 and 2 refer to the media in which the beams are and + and - refer to beams moving towards and away from the interface respectively. What are the coefficients of the 2×2 matrix M in terms of r and t?

(c) Now, consider three media 1, 2 and 3. 1 and 3 are semi-infinite extending out from the interfaces they share with 2. The medium 2 sandwiched between 1 and 3 is of thickness d. The refractive indices of the three media are n_1, n_2 and n_3 respectively and the frequency of light is ω. If a beam of light of amplitude E is incident onto the interface between 1 and 2, what is the amplitude of the emergent beam in medium 3? The refection and transmission coefficients between 1 and 2 are r_{12} and t_{12}, while those between 2 and 3 are r_{23} and t_{23}.

(*Hint: You do not need to use detailed forms of the Fresnel equations or boundary conditions for the fields for any part of this problem.*)