PH 206: Electromagnetic theory

Problem Set 7

1. Prove the continuity equation for charge

$$\boldsymbol{\nabla}.\mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

2. Consider two coordinate systems O and O' in which Maxwell's equations hold. The space and time coordinates in O are x, y, z and t while the charge density, current density, electric field and magnetic field are $\rho(\mathbf{r}, t)$, $\mathbf{J}(\mathbf{r}, t)$, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ respectively. The corresponding quantities in O' are x', y', z', t', $\rho'(\mathbf{r}', t')$, $\mathbf{J}'(\mathbf{r}', t')$, $\mathbf{E}'(\mathbf{r}', t')$ and $\mathbf{B}'(\mathbf{r}', t')$. In each of the following cases, work out how the charge density, current density, electric field and magnetic field in the two frames are related to one another.

   (a) O and O' are related by a spatial translation, i.e. $\mathbf{r}' = \mathbf{r} + \mathbf{d}$ and $\mathbf{d}$ is a constant.

   (b) O and O' are related by a temporal translation, i.e. $t' = t + T$ and T is a constant.

   (c) O and O' are related by a spatial rotation, i.e.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

   where R is a 3D rotation matrix.

   (d) O and O' are related by a spatial inversion (parity transformation), i.e. $\mathbf{r}' = -\mathbf{r}$.

   (e) O and O' are related by a temporal inversion, i.e. $t' = -t$.

   (f) O and O' are related by charge conjugation, i.e. the signs of all charges in O' are reversed compared to O

3. The Hall effect involves applying a magnetic field $\mathbf{B}$ to a material which has a current of density $\mathbf{J}$ flowing in it. This usually causes an electric field $\mathbf{E}$ to develop in the system. $\mathbf{E}$ is related to $\mathbf{J}$ by a resistivity tensor $\rho_{\alpha\beta}(\mathbf{B})$ as $E_\alpha = \rho_{\alpha\beta}(\mathbf{B})J_\beta$. For a spatially homogeneous and and isotropic material argue on the basis of spatial rotational, translational and inversion symmetry that the conductivity tensor has the general form

$$\rho_{\alpha\beta} = P\delta_{\alpha\beta} + Q\epsilon_{\alpha\beta\gamma}B_\gamma + R\delta_{\alpha\beta}B_\gamma B_\gamma + SB_\alpha B_\beta,$$

   to second order in $\mathbf{B}$ about $\mathbf{B} = 0$, where P, Q, R and S are constants. What information, if any, can charge conjugation and temporal inversion give you? Remember that $\mathbf{B}$ is an externally applied filed and is thus not related to the current $\mathbf{J}$ in any way, which itself is caused to flow in the material by an external source.

4. Consider a collection of point masses $\{m_i\}$ interacting through a pairwise mechanical potential $U_m(|\mathbf{r}_i - \mathbf{r}_j|)$.

   (a) If $\mathbf{r}_i(t)$ is the position of the i^{th} particle at time t, what is an appropriate expression for the energy density U_{mech} of the system?

   (b) Assuming the particles obey Newton's laws of motion, show that there is a continuity equation

$$\frac{\partial U_{mech}}{\partial t} = -\boldsymbol{\nabla}.\mathbf{S}_{mech},$$

   where $\mathbf{S}_{mech}$ is a mechanical "Poynting vector" or energy flux. What is the expression for $\mathbf{S}_{mech}$?

   (c) Now, assume that the particles are charged and show that total energy conservation implies

$$\frac{\partial}{\partial t}\left(U_{mech} + U_{em}\right) = -\boldsymbol{\nabla}.\left(\mathbf{S}_{mech} + \mathbf{S}_{em}\right),$$

   where U_{em} and $\mathbf{S}_{em}$ are the electromagnetic energy density and Poynting vector respectively.

5. If magnetic monopoles existed, the magnetic field due to a monopole of strength g would be given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r},$$

in complete analogy with the expression for the electric field of a point charge. Assume that there is a magnetic monopole of strength g separated by a distance d from a point charge q. The monopole gives rise to a magnetic field given by the above expression while the point charge gives rise to an electric field. The electric field and magnetic field produce an angular momentum given by the expression

$$\mathbf{L} = \epsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{r},$$

where the integral is over all space.

(a) Show that the total momentum carried by the fields is zero and thus the above expression for $\mathbf{L}$ is independent of choice of origin. You can do this without explicitly calculating the relevant integral but instead by observing its form and using symmetry arguments.

(b) In which direction is the angular momentum? Once again, you do not need to explicitly evaluate the integral and only need to use arguments of symmetry.

(c) Now, evaluate the integral and obtain the value of $\mathbf{L}$. Does it depend on d?

(d) Suppose the angular momentum is quantized in units of $\hbar$. Show that for a given g, this implies that the charge q is also quantized.