PH 206: Electromagnetic theory

Problem Set 5

1. (a) Show that the magnetic field $\mathbf{B}(\mathbf{r})$ due to a loop of arbitrary shape carrying a current I is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\Omega(\mathbf{r}),$$

where $\Omega(\mathbf{r})$ is the solid angle subtended by the loop at the point $\mathbf{r}$. Start with the Biot-Savart law for the magnetic field due to a current carrying loop and recall that the solid angle subtended by an area element $d\mathbf{S}$ is given by

$$d\Omega = \frac{d\mathbf{S}.\hat{r}}{r^2},$$

where $\mathbf{r}$ is the vector from the area element to the point at which the solid angle is subtended.

(b) Now, consider an infinitely long right cylindrical solenoid of arbitrary cross section with its axis along $\hat{z}$. Imagine that the solenoid has been created by stacking coils of the shape of the cross section, each carrying a current I. Assume that the number of coils per unit length along the axis is N and they are sufficiently dense that you can assume that the solenoid is a current carrying sheet. Use the result of (a) to show that the field B at any point inside the solenoid is equal to

$$\mathbf{B} = \mu_0 N I\hat{z}.$$

Also show that the magnetic field anywhere outside the solenoid is zero. (*Hint: Recall that the total solid angle subtended by a closed surface at point is 4π if the point is enclosed by the surface and zero otherwise. Also, since the solenoid is infinitely long, you can consider the point O at which the field has to be calculated to be in the plane $z = 0$ without loss of generality and with coordinates x' and y'. Let $\Omega(x', y', z)$ be the solid angle subtended at O by the coil at coordinate z. What is the relation between $\Omega(x', y', z)$ and $\Omega(x', y', -z)$? Use this to obtain the required result.*)

2. In this problem you will investigate the uniqueness of the magnetic field $\mathbf{B}$ and magnetic vector potential $\mathbf{A}$. Imagine that the usual equations of magnetostatics $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J}$ and $\boldsymbol{\nabla}.\mathbf{B} = 0$ are specified in a volume Ω with given current density $\mathbf{J}$. The surface of Ω is $\partial\Omega$

(a) Show that the magnetic field is uniquely determined by specifying the normal component of $\mathbf{B}$ on $\partial\Omega$.

(b) Show that the magnetic field can also be uniquely determined by either specifying the tangential component of $\mathbf{B}$ or the tangential component of $\mathbf{A}$ on $\partial\Omega$.

(c) The conditions in (b) determine $\mathbf{B}$ uniquely but not $\mathbf{A}$. Show that $\mathbf{A}$ can be determined uniquely if in addition the normal component of $\mathbf{A}$ is specified on $\partial\Omega$. (*Hint: Assume two solutions $\mathbf{A}_1$ and $\mathbf{A}_2$ with corresponding fields $\mathbf{B}_1$ and $\mathbf{B}_2$. Define $\delta\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2$ and $\delta\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2$. Consider $\int_\Omega \delta\mathbf{A}.(\boldsymbol{\nabla}\times\delta\mathbf{B})\ d^3\mathbf{r}$. Why is this quantity zero? Using vector identities show that this implies that $\boldsymbol{\nabla}\times\delta\mathbf{A} = 0$ and $\boldsymbol{\nabla}.\delta\mathbf{A} = 0$ everywhere in Ω.*)

3. Consider two identical circular loops of radius R. One of them has its centre at $z = a$ and the other at $z = -a$ and both are oriented perpendicular to the z axis. Further, both of them carry the same current I flowing in the same direction. This direction can be assumed to be counterclockwise looking down at both loops. You will calculate the magnetic field $\mathbf{B}(\rho, z)$ close to the origin in this problem, where ρ and z are cylindrical polar coordinates. There is no dependence on the angle coordinate ϕ because of axial symmetry.

(a) Calculate the magnetic field at the origin $\mathbf{B}(0, 0)$.

(b) Show that $B_\phi(\rho, z)$ is zero everywhere. (*Hint: Consider the Taylor expansion of B_ϕ about the origin and argue that all derivatives are zero.*)

(c) Calculate $\mathbf{B}(\rho, z) - \mathbf{B}(0, 0)$ to lowest order in ρ and z.

(d) Now, assume that the currents carried by the two loops are in opposite senses. The one at $z = a$ carries a counterclockwise current looking down at it and the one at $z = -a$, a clockwise current. Work out (a), (b) and (c) for this case.

(e) Is it possible for the lowest order correction to $\mathbf{B}(\rho, z) - \mathbf{B}(0,0)$ to be zero in both cases? If so, what is the condition on a and R for this to happen?

4. Prove that

$$\int (\mathbf{r}.\mathbf{r}')\,\mathbf{J}(\mathbf{r}')d\mathbf{r}' = -\int \mathbf{r}'\,[\mathbf{r}.\mathbf{J}(\mathbf{r}')]\,d\mathbf{r}',$$

for a current density $\mathbf{J}(\mathbf{r}')$ localized to a volume which producing a static magnetic field. The integrals are over all space. Recall that this is the relation we had used to derive the magnetic dipole term in the multipole expansion for the magnetic vector potential.

5. In this problem, you will obtain the multipole expansion for magnetostatics using a scalar potential instead of a vector potential. Consider a current distribution $\mathbf{J}(\mathbf{r}')$ which is confined to a finite region Ω of space.

(a) Show that

$$\boldsymbol{\nabla}^2\,(\mathbf{r}.\mathbf{B}) = -\mu_0 \mathbf{r}.\boldsymbol{\nabla}\times\mathbf{J}.$$

(b) Perform a multipole expansion for the quantity $\mathbf{r}.\mathbf{B}$ starting from the above equation just as you do for the potential in electrostatics.

(c) We wish to evaluate the magnetic field $\mathbf{B}(\mathbf{r})$ due $\mathbf{J}$ at a point $\mathbf{r}$ outside Ω. Since the current density at $\mathbf{r}$ is zero, we have

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= 0, \\ \boldsymbol{\nabla}.\mathbf{B} &= 0, \end{aligned}$$

at $\mathbf{r}$. We can thus define a scalar potential ϕ_m outside Ω such that $\mathbf{B} = -\boldsymbol{\nabla}\phi_m$. What is $\mathbf{r}.\mathbf{B}$ in terms of ϕ_m? Obtain the multipole expansion for ϕ_m from (b) analogous to the what you obtain for the potential in electrostatics. What is the effective "charge density" in this multipole expansion?

(d) Show that the monopole moment is zero and the dipole moment is given by the usual expression for magnetostatics obtained in class. Obtain an expression for the quadrupole moment and verify that it is a symmetric traceless second rank tensor.