PH 206: Electromagnetic theory

Problem Set 4

1. While calculating the energy of a charge distribution $\rho(\mathbf{r})$ in an external electric field, it was shown that the 2^l-pole moment $M^{(l)}$ of the distribution has the following form

$$M^{(l)}_{i_1 i_2 \ldots i_l} = \int_\Omega x_{i_1} x_{i_2} \ldots x_{i_l}\ \rho(\mathbf{r}) d^3\mathbf{r} - A^{(l)}_{i_1 i_2 \ldots i_l},$$

where $i_1, i_2 \ldots i_l$ take the values 1,2 and 3 with $x_1 = x$, $x_2 = y$ and $x_3 = z$. The tensor $A^{(l)}$ makes the 2^l-pole moment $M^{(l)}$ traceless. Further, $M^{(l)}$ is symmetric under the interchange of any two indices and so is $A^{(l)}$. Show that $M^{(l)}$ has $2l + 1$ independent components. It might help to think of the components of $M^{(l)}$ to be of the form $\int_\Omega x^p y^q z^r \rho(\mathbf{r}) d^3\mathbf{r}$ plus a part to make the tensor traceless. Here p, q and r are non-negative integers whose sum is l.

2. In class it was asserted that the total energy stored in a distribution of free and bound charges is equal to the work done to bring the *free charges alone* from infinity to their final positions. In this problem, you will prove this result. Consider a system of free charges at positions $\{\mathbf{r}_i\}$ and bound charges at positions $\{\mathbf{R}_\alpha\}$, where i runs over all the free charges in the system and α over all the bound charges. Two free charges i and j interact with one another through the Coulomb potential $U_c(\mathbf{r}_i - \mathbf{r}_j)$. Similarly a free charge i and bound charge α also interact via the Coulomb potential $U_c(\mathbf{r}_i - \mathbf{R}_\alpha)$. The interaction between two bound charges α and β is via a potential $U_b(\mathbf{R}_\alpha - \mathbf{R}_\beta)$, which is a sum of the Coulomb potential and a mechanical potential that arises from the mechanical interaction between the bound charges. As the free charges are brought from infinity to their positions $\{\mathbf{r}_i\}$, the bound charges rearrange themselves so that the net force on them is zero. Thus, the positions $\{\mathbf{R}_\alpha\}$, of the bound charges are functions of the positions $\{\mathbf{r}_i\}$ of the free charges.

   (a) For a given set of positions $\{\mathbf{r}_i\}$ and $\{\mathbf{R}_\alpha\}$, what is the total energy of the system?

   (b) Now, consider the i^{th} free charge as it is brought from infinity to $\mathbf{r}_i$. What is the force $\mathbf{F}_j(\mathbf{r}'_i)$ on this free charge when it is at a position $\mathbf{r}'_i$ due to the free charge at position $\mathbf{r}_j$? Express the force in terms of the derivative of $U_c(\mathbf{r}_i - \mathbf{r}_j)$. What is the force $\mathbf{F}_\alpha(\mathbf{r}'_i)$ due to the bound charge at position $\mathbf{R}_\alpha$? Express this force too in terms of the derivative of the potential $U_c(\mathbf{r}_i - \mathbf{R}_\alpha)$. Be careful since $\mathbf{R}_\alpha$ is itself a function of $\mathbf{r}'_i$.

   (c) The work done to bring the i^{th} charge to $\mathbf{r}_i$ is equal to

$$W_i = -\int_\infty^{\mathbf{r}_i} \left(\sum_{j \neq i} \mathbf{F}_j(\mathbf{r}'_i) + \sum_\alpha \mathbf{F}_\alpha(\mathbf{r}'_i) \right) .d\mathbf{r}'_i.$$

   Show that this work is equal to the difference in total energy of the system between having the i^{th} free charge at infinity and having it at $\mathbf{r}_i$. *(Hint: Use the fact that the force on every bound charge is zero at all times during the process of bringing the i^{th} charge to $\mathbf{r}_i$.)* In this way, the sum of W_i over all free charges is equal to the total energy you calculated in (a).

3. The Van der Waals interaction between molecules is electrostatic in nature. Assume that the molecules have polarizability α, i.e. upon the application of an electric field $\mathbf{E}$, the induced polarization of the molecule is $\mathbf{p} = \alpha\mathbf{E}$. The molecules have no permanent electric dipole moment. The interaction between two molecules is caused by one developing a dipole moment momentarily due to a fluctuation and inducing a dipole moment in the other a distance r away. The two dipole moments then interact with each other. The average dipole moment over all fluctuations is zero but the mean square moment is not. Further, the molecules are completely isotropic in space as are the fluctuations. What is the Van der Waals interaction potential as a function of the distance r? Is the resultant force attractive or repulsive?

4. In this problem you will calculate the field produced due to a dielectric sphere of radius R in two different situations.

(a) First consider a dielectric sphere of permittivity ϵ which is placed in a uniform electric field $\mathbf{E}_0$, i.e. had the sphere not been present the field in all of space would have been $\mathbf{E}_0$. Calculate the field $\mathbf{E}$, the displacement vector $\mathbf{D}$ and polarization density $\mathbf{P}$ everywhere. What is the total induced dipole moment in the sphere and what are the volume and surface bound charge densities?

(b) Now, assume that there is no applied electric field but the sphere has a uniform polarization density "frozen in" such that the total dipole moment is the same as in the previous part. Calculate $\mathbf{E}$, $\mathbf{D}$ and the volume and surface bound charge densities.

5. In this problem you will derive expressions for the work done in setting up dipole moments in a dielectric. Consider a parallel place capacitor connected to a battery so that there is an electric field $\mathbf{E_0}$ between the plates. A dielectric slab of volume V is introduced between the two plates which causes it to develop a uniform polarization density $\mathbf{P}$. Show that the work done by the battery to change the polarization by an amount $d\mathbf{P}$ is $dW = \mathbf{E_0}.d\mathbf{P}V$. (*Hint: If the polarization of the slab changes, it will try to change the potential difference* $\Delta\phi$ *between the plates. However,* $\Delta\phi$ *cannot change since the plates are connected to a battery of a fixed voltage. The only way* $\Delta\phi$ *can remain constant is if the charge on the plates changes to counter the effect of the changing polarization. Thus the work done by the battery is the work done to supply the extra charge to the plates.*)