PH 206: Electromagnetic theory

Problem Set 3

1. Consider Poisson's equation in one dimension

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

with the boundary condition that $V(a) = V(-a) = 0$.

   (a) Calculate the Green's function for the Laplacian operator for the above problem analytically.

   (b) Now, discretize the Laplacian operator so that it is a matrix. Calculate the Green's function numerically by inverting this matrix and check that it matches the analytical solution. Pick an appropriate lattice size for the numerical calculation.

2. Now consider Poisson's equation in 1D but on a ring of perimeter $2a$, i.e. with periodic boundary conditions $V(a) = V(-a)$.

   (a) Calculate the Green's function for the Laplacian operator in this geometry analytically.

   (b) Again, discretize the Laplacian operator so that it is a matrix and calculate its Green's function numerically. Check it against the analytical solution.

Did you encounter a problem with calculating the Green's function in this geometry? What do you think is the origin of the problem and why does it not occur in the previous case?

3. Consider Poisson's equation

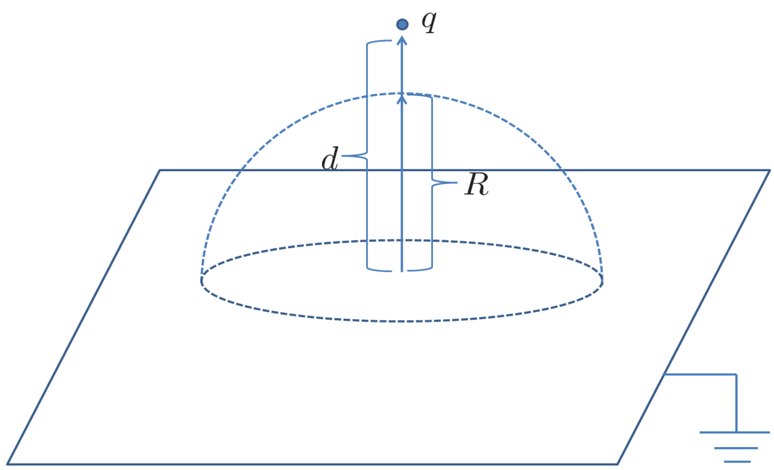
$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

in d dimensions with $d > 2$ with the boundary condition that $V(r \to \infty) \to 0$. Also assume that space is homogeneous. Show that the Green's function for the Laplacian operator

$$G(\mathbf{r}, \mathbf{r}') = A_d \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)^{d-2},$$

where A_d is a numerical constant. You do not have to calculate the value of A_d.

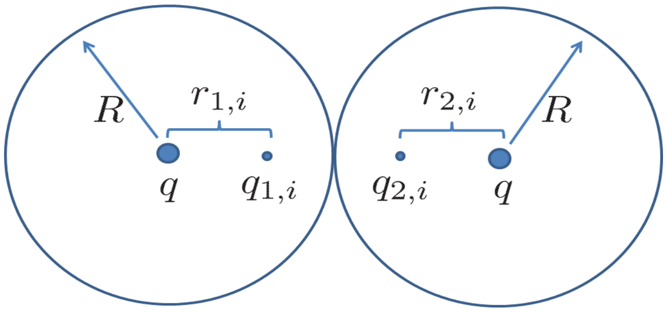
4. An infinite grounded conducting plane has a hemispherical bump of radius R. A point charge q is placed at a distance d above the centre of the hemisphere as shown in the figure (the line joining the charge and the centre of the hemisphere is perpendicular to the conducting plane).



(a) Calculate the electric field $\mathbf{E}(\mathbf{r})$ everywhere.

(b) What is the total charge on the conducting plane (with the bump)?

5. In this problem, using the method of images you will show that the capacitance of two identical metallic spheres of radius R in contact with each other is $C = 4\pi\epsilon_0 R \ln 4$. To do this assume that the two spheres (call them 1 and 2) are in contact and each has a point charge q at its centre. The surface of neither sphere is an equipotential due to the point charge at the centre of the other. To make 1 an equipotential, introduce an image charge in 1 for the charge at the centre of 2. Now, 2 is not an equipotential so introduce two image charges in 2, one for q at the centre of 1 and the other for the image charge you introduced in 1. 1 is now not an equipotential so introduce two more image charges in 1 corresponding to the image charges you introduced in 2 and keep going this way ad infinitum. Let $q_{1,i}$ and $q_{2,i}$ be the i^{th} image charges introduced in spheres 1 and 2 respectively. Also, let $r_{1,i}$ and $r_{2,i}$ be the distances of these image charges from the centres of spheres 1 and 2 respectively. $q_{1,0} = q_{2,0} = q$ and $r_{1,0} = r_{2,0} = 0$.



(a) Obtain recursion relations among $q_{1,i}$, $q_{2,i}$, $r_{1,i}$ and $r_{2,i}$.

(b) Calculate the total charge contained in the system q_{tot} in terms of q.

(c) Calculate the potential V on the surface of the touching spheres. Be careful while performing the infinite sum of the potentials due to all the charges.

(d) Show that $q_{tot}/V = C = 4\pi\epsilon_0 R \ln 4$.