PH 206: Electromagnetic theory

Problem Set 2

1. In this problem you will figure out the effect of the curvature of the surface of a conductor on the electric field just above it. Consider three different conductors; 1) an infinite sheet, 2) an infinite cylinder of radius R_c and 3) a sphere of radius R_s. All of them have a uniform surface charge density σ. From the discussion in class, it is clear that the electric field E at any point just above the surface of any of the three conductors is σ/ϵ_0 and normal to the surface.

   (a) Calculate the normal derivative $\frac{\partial E}{\partial n}$ just above the surface in each case and show that it depends on the shape of the conductor.

   (b) Show that in general at a point just above the surface of a conductor,

   $$\frac{\partial E}{\partial n} = -E\left(\frac{1}{R_1} + \frac{1}{R_2}\right),$$

   where R_1 and R_2 are the principle radii of curvature at the surface. (*Hint: Use the fact that* $\boldsymbol{\nabla}.\mathbf{E} = 0$ *just above the surface.*)

   (c) Verify that $\frac{\partial E}{\partial n}$ that you have obtained in (b) matches what you have obtained in (a) for the three cases.

   (d) Is the result derived in (b) valid only for conductors or any object whose surface is an equipotential?

2. Show that for a potential that satisfies Laplace's equation everywhere inside a sphere, its value at the origin is equal to its average over the surface of the sphere. (*Hint:* $V(0) = \int_{sphere} V(\mathbf{r})\delta(\mathbf{r})\ d^3\mathbf{r}$ *and* $\boldsymbol{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\mathbf{r})$.)

3. Consider a set of N conductors such that the i^{th} conductor with surface $\mathbb{S}_i$ has charge Q_i. There are no other charges.

   (a) Show that the electrostatic energy of the configuration is

   $$\epsilon = \frac{\epsilon_0}{2}\int_{\Omega'} |\mathbf{E}|^2\ d^3\mathbf{r},$$

   where Ω' is the volume of all space excluding the volumes of the conductors.

   (b) Now, assume that the charges on each surface $\mathbb{S}_i$ are arranged in such a way that it is not an equipotential. Show that this yields a a larger value of ϵ, where ϵ is given in the previous part. (*Hint: Assume that the new configuration corresponds to an electric field* $\mathbf{E}'$ *in* Ω'. *Show that* $\Delta\epsilon = \int_{\Omega'} \left(|\mathbf{E}'|^2 - |\mathbf{E}|^2\right) d^3\mathbf{r} > 0$ *using vector identities to manipulate the integrand.*)

4. A balloon made of a conducting material of surface tension T is initially completely deflated (has zero volume). The balloon is then slowly charged until the charge on it is q. Assume that the balloon is always spherical during the charging process.

   (a) What is the radius of the balloon after it has been fully charged?

   (b) What is the total electrostatic energy ϵ calculated according to the relations

   $$\epsilon = \frac{\epsilon_0}{2}\int_{allspace} \mathbf{E}^2 d\mathbf{r}$$

   and

   $$\epsilon = \frac{1}{2}\int \rho V d\mathbf{r},$$

   where the symbols have their usual meanings? Does the energy obtained from the two expressions agree with each other?

   (c) What is the total work done W to charge the balloon?

(d) Does W obtained in part (c) agree with the values of ϵ obtained in part (b)? If not, how will you resolve the discrepancy?

5. A cube of side a with no charges inside has two opposite faces at potentials V_1 and V_2 and all other faces at zero potential. Calculate the potential everywhere inside the cube.

6. The northern and southern hemispheres of a spherical shell have uniform surface charge density σ and -σ respectively. Let $V(r, \theta, \phi)$ in spherical polar coordinates with the origin at the origin of the sphere.

   (a) What is the leading order term in the potential as $r \to \infty$? Calculate the first three sub-leading order terms.

   (b) What is the leading order term in the potential as $r \to 0$? Calculate the first three sub-leading order terms.