PH 206: Electromagnetic theory

Problem Set 9

1. The entire xy plane has uniform surface charge density σ in the unprimed frame of reference, which has coordinates $(\mathbf{r}, t)$. This frame moves with velocity $\mathbf{v} = v\hat{z}$ with respect to the primed frame which has coordinates $(\mathbf{r}', t)$. The two sets of axes coincide at $t = t' = 0$.
   (a) What are the electric fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ in the unprimed frame?
   (b) What are the charge density $\rho'(\mathbf{r}', t')$, current density $\mathbf{J}'(\mathbf{r}', t')$, electric field $\mathbf{E}'(\mathbf{r}', t')$ and magnetic field $\mathbf{B}'(\mathbf{r}', t')$ in the primed frame?
   (c) Verify that the quantities calculated in (b) satisfy the Maxwell equations.

2. The non-relativistic Abraham-Lorentz formula for the radiation reaction as obtained in class is
$$\mathbf{F}_{\text{rad}} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\dot{\mathbf{a}}.$$
   A plausible relativistic generalization of this formula appears to be
$$K^\mu = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\frac{da^\mu}{d\tau},$$
   where K^μ is the Minkowski force for the radiation reaction and a^μ is the four acceleration. The above formula is obviously relativistically covariant and reproduces the proper non-relativistic result for $\mathbf{F}_{\text{rad}}$ in the particle's rest frame.
   (a) Show that the above formula for K^μ is actually not correct since it does not give the right expression for K^0 in the particle's rest frame. One way to see this is to evaluate the Lorentz scalar $a^\mu U_\mu$, where U^μ is the four velocity and remembering that $K^\mu = m_0 a^\mu$.
   (b) The formula for K^μ can be fixed by adding an additional term to it so that we obtain all the components of K^μ in the particle's rest frame correctly. Clearly, this term has to be a four vector Λ^μ for the resultant expression to also be relativistically covariant. Deduce the form of Λ^μ.

3. The free-field Lagrangian density for the EM field is
$$\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}.$$
   The canonical stress-energy tensor can be obtained from $\mathcal{L}$ using the expression
$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)}\partial^\nu A^\lambda - g^{\mu\nu}\mathcal{L}.$$
   (a) Evaluate T^{00} and show that it is equal to the energy density plus a term that goes to zero when integrated over all space for a localized charge and current distribution.
   (b) Evaluate T^{0i}, where $i = 1, 2$ or 3 and show that it is equal to the i^{th} component of the Poynting vector plus a term that goes to zero when integrated over all space for a localized charge and current distribution.
   (c) What can you say about T^{ij}, where i and j both take values that go from 1 to 3.
   (d) Show that $\partial_\mu T^{\mu\nu} = 0$.

4. Consider a metal which has free electrons. Ohm's law is given by the expression
$$\mathbf{J} = \sigma\mathbf{E},$$
   where $\mathbf{J}$ is the current density, $\mathbf{E}$, the electric field and σ, the electrical conductivity. The above expression for $\mathbf{J}$ holds only in the rest frame of the metal. If the metal moves with a velocity $\mathbf{v}$, the non-relativistic ($v \ll c$) generalization is
$$\mathbf{J} = \sigma\mathbf{E} + \rho\mathbf{v},$$

where ρ is the density of free electrons. The additional term comes from the fact that a moving charge distribution causes a current even if there is no electric field to drive it. In this problem, you will obtain the relativistic form of Ohm's law.

(a) A naive covariant generalization of Ohm's law is

$$J^\mu = \frac{\sigma}{c} F^{\mu\nu} U_\nu,$$

where U_ν is the four velocity of the medium. Show that this produces the correct form of Ohm's law in the rest frame of the metal but not the non-relativistic version in a moving frame. Also show that it does not give a sensible value for ρ in the rest frame of the metal.

(b) Both the above problems can be fixed by adding an extra "convective" term to the right hand side of the covariant equation in a). Argue that this term has the form $\frac{1}{c^2}\left(U_\nu J^\nu\right) U^\mu$, so that the correct equation is

$$J^\mu = \frac{\sigma}{c} F^{\mu\nu} U_\nu + \frac{1}{c^2}\left(U_\nu J^\nu\right) U^\mu.$$

Show that this equation yields the right expressions for J and ρ in the non-relativistic limit.

(c) The metal moves with a velocity $\mathbf{v}$ with respect to some inertial frame of reference. If $\mathbf{E}$, $\mathbf{B}$ and ρ are the electric field, magnetic field and charge density respectively in this frame of reference, find an expression for the current density $\mathbf{J}$.

5. Assume that magnetic charges exist along with associated currents. Maxwell's equations would then have to be modified to look like

$$\begin{aligned}
\boldsymbol{\nabla}.\mathbf{E} &= \frac{\rho_e}{\epsilon_0}, \\
\boldsymbol{\nabla}\times\mathbf{E} &= -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}, \\
\boldsymbol{\nabla}.\mathbf{B} &= \mu_0 \rho_m, \\
\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\left(\mathbf{J}_e + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right),
\end{aligned}$$

where ρ_e, $\mathbf{J}_e$ and ρ_m, $\mathbf{J}_m$ are the electric charge density, current density and magnetic charge density, current density respectively.

(a) Define a four electric current j_e^μ and four magnetic current j_m^μ. Show that the modified Maxwell equations given above can be written as

$$\begin{aligned}
\partial_\mu G^{\mu\nu} &= \mu_0 J_e^\nu, \\
\partial_\mu \mathcal{G}^{\mu\nu} &= -\mu_0 J_m^\nu,
\end{aligned}$$

where $G^{\mu\nu}$ is analogous to $F^{\mu\nu}$ for regular electrodynamics and $\mathcal{G}^{\mu\nu}$ is the dual of $G^{\mu\nu}$. What is $G^{\mu\nu}$ in terms of the fields $\mathbf{E}$ and $\mathbf{B}$? What about $\mathcal{G}^{\mu\nu}$?

(b) Since $\partial_\mu \mathcal{G}^{\mu\nu} \neq 0$, there is no four potential A^μ such that $G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. One can however define electric and magnetic four potentials A_e^μ and A_m^μ such that

$$\begin{aligned}
\partial_\mu F_e^{\mu\nu} &= \mu_0 J_e^\nu, \\
\partial_\mu \mathcal{F}_e^{\mu\nu} &= 0, \\
\partial_\mu F_m^{\mu\nu} &= -\mu_0 J_m^\nu, \\
\partial_\mu \mathcal{F}_m^{\mu\nu} &= 0,
\end{aligned}$$

where $F_e^{\mu\nu} = \partial^\mu A_e^\nu - \partial^\nu A_e^\mu$ and $F_m^{\mu\nu} = \partial^\mu A_m^\nu - \partial^\nu A_m^\mu$. Further, $\mathcal{F}_e^{\mu\nu}$ is the dual of $F_e^{\mu\nu}$ and $\mathcal{F}_m^{\mu\nu}$ is the dual of $F_m^{\mu\nu}$. What are $G^{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ in terms of the quantities defined above?

(c) Write down the Lagrangian that will yield the Maxwell equations for $\mathbf{E}$ and $\mathbf{B}$ given above.

(d) Show that the invariance of the Lagrangian under gauge transformations of A_e^μ and A_m^μ result in conservation laws for the electric and magnetic charge respectively.