PH 206: Electromagnetic theory

Problem Set 1

1. A vector field $\mathbf{F}(\mathbf{r})$ obeys $\boldsymbol{\nabla} \times \mathbf{F} = 0$ in a volume Ω. Show that this implies that there exists a scalar field $\phi(\mathbf{r})$ such that $\mathbf{F} = -\boldsymbol{\nabla}\phi$ in Ω. It might help to guess a form for ϕ and then show that it satisfies $\mathbf{F} = -\boldsymbol{\nabla}\phi$.

2. Now suppose that $\mathbf{F}(\mathbf{r})$ obeys $\boldsymbol{\nabla}.\mathbf{F} = 0$ in Ω. Show that this implies that there exists a vector field $\mathbf{A}$ such that $\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{A}$ in Ω. As in the previous part, it might help to guess a form for $\mathbf{A}$ and then show that it satisfies $\mathbf{F} = -\boldsymbol{\nabla} \times \mathbf{A}$.

3. Consider the two dimensional surface $\mathbb{S} = \mathbb{R}^2 - (0,0)$, which is just the entire two dimensional plane with the origin removed. Define $\phi(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

   (a) Show that it is not possible for $\phi(x,y)$ to be single valued and for $\boldsymbol{\nabla}\phi$ to exist everywhere.

   (b) Suppose $\boldsymbol{\nabla}\phi$ exists everywhere on $\mathbb{S}$. What is an appropriate choice of multi-valued function $\phi(x,y)$ for this to happen. What is $\boldsymbol{\nabla}\phi$? (*Hint: Think in terms of polar co-ordinates.*)

   (c) Let $\mathbf{F} = -\boldsymbol{\nabla}\phi$ for ϕ from the previous part. What is $\boldsymbol{\nabla} \times \mathbf{F}$?

   (d) Calculate the line integral

   $$\oint \mathbf{F}.d\mathbf{l}$$

   along a circle with origin in $\mathbb{S}$ with origin at (0,0). Does your result seem to contradict Stokes' theorem

   $$\oint \mathbf{F}.d\mathbf{l} = \iint \boldsymbol{\nabla} \times \mathbf{F}.d\mathbf{S},$$

   where the surface integral is over a region bounded by the circle. If so, how will you fix Stokes' theorem?

   (e) Now let $\mathbb{S} = \mathbb{R}^2$, i.e. the xy plane including the origin and $\phi(x,y)$ is the same as in (b). Is there a contradiction of Stokes' theorem? If so, how will you fix it? What is $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi$?

4. A constant charge density ρ fills *all of space.*

   (a) What might you expect the value of the electric field $\mathbf{E}(\mathbf{r})$ at a point $\mathbf{r}$ to be from considerations of "symmetry"?

   (b) Does the value of the electric field you obtained in the previous part obey Gauss's law $\boldsymbol{\nabla}.\mathbf{E}(\mathbf{r}) = \rho/\epsilon_0$? What do you think is the $\mathbf{E}(\mathbf{r})$ that satisfies Gauss's law?

   (c) What value of $\mathbf{E}(\mathbf{r})$ will you obtain if you calculate it from the integral

   $$\mathbf{E}(\mathbf{r}) = \frac{\rho}{4\pi\epsilon_0}\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'?$$

   Does this agree with what you get in parts (a) and (b)? If not, how do you reconcile the different values with one another?

5. Consider a set of N conductors. The charge and potential of the i^{th} conductor are Q_i and V_i respectively. If there are no other charges anywhere and the potential at ∞ is zero, show that

   $$V_i = \sum_{j=1}^{N} B_{ij}Q_j,$$

   where B_{ij} are coefficients that depend only on the geometries and arrangements of the conductors. In general, does B_{ij} depend only on the geometries and positions of conductors i and j or the entire arrangement of the N conductors?

6. The relation between the charges and potentials of the conductors in the previous problem can be inverted to give

$$Q_i = \sum_j C_{ij} V_j,$$

where C_{ij} are the coefficients of capacitance. Show that $C_{ij} = C_{ji}$ and $B_{ij} = B_{ji}$.