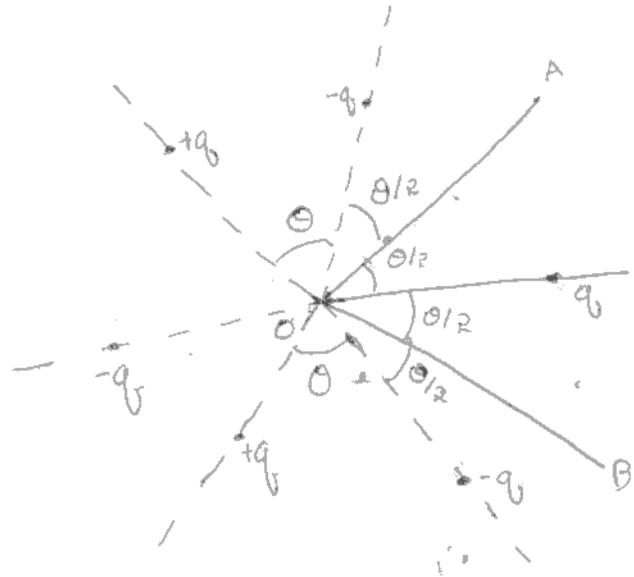
PH R06

MIDTERM EXAM SOLUTIONS

1)



a) & b) The positions of the image charges can be determined as follows:

An image charge $-q$ has to be placed at angle $\theta/2$ behind OA to make it a zero potential equipotential.

Similarly an image charge $-q$ has to be placed $\theta/2$ behind OB to make it a zero potential equipotential.

Thus the first pair of image charges have charge and position

$-q \quad (d, \theta)$ and $-q \quad (d, 2\pi - \theta)$ where the positions are in polar coordinates.

Neither OA nor OB is an equipotential after these two image charges have been put in place.

Thus, we need to add the image charge for the $-q, (d, \theta)$ image charge behind OB to make it an equipotential. The charge and coordinates for this are $+q, (d, 2\pi - 2\theta)$.

Similarly the image charge of the image charge $-q\,(d, 2\pi-\theta)$ behind OA has charge and position $+q$, $(d, 2\theta)$.

However ~~now~~ neither OA nor OB is an equipotential anymore, so we need another pair of image charges and need to keep going this way.

~~AB~~ At the m^{th} stage, the pair of image charges have charge and position $(-1)^m q$, $(d, m\theta)$ & $(-1)^m (d, 2\pi - m\theta)$

If the image solution has a finite number of charges, eventually for some m $(d, m\theta) = (d, 2\pi - m\theta)$ or $\theta = \frac{\pi}{m}$. The last image charge is thus at $\theta = \pi$, diametrically opposite the ~~charge~~ original charge q. The total number of image charges is $2m-1 = m$, which is an odd number. The angle

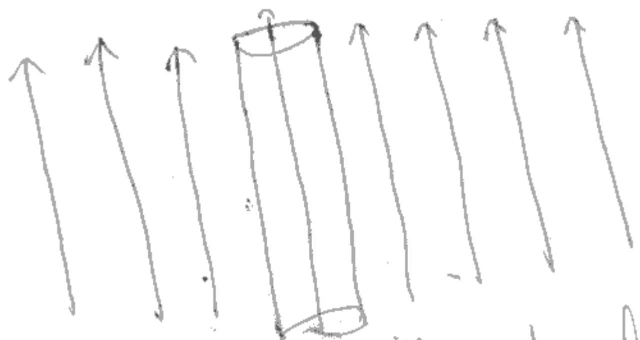
$$\theta = \frac{\pi}{m} = \frac{2\pi}{m+1}.$$

The co-ordinates of the image charges are $(d, m\theta)$ and $(d, 2\pi - m\theta)$. The original charge is at $m=0$ and the final image charge is at $\theta = \pi$. Thus, the image charges along with the original charge are at the vertices of ~~a m+1~~ regular polygon centred at the origin with $n+1$ vertices.

The total induced charge over both planes is the sum of all the image charges. It is easy to see that this is always $= -q$. Since the charge is distributed evenly on both planes from the symmetry of the problem, the charge on each plane is thus $= -q/2$.

1.c) If the two planes are parallel, $\theta = 0 \Rightarrow \frac{2\pi}{n+1} \to 0$ or $n \to \infty$.
Further, the "point of intersection" of the two parallel planes has to be at an infinite distance from the charge q so $d \to \infty$.
However, the perpendicular distance from either plane to the x axis is $d \sin\theta = d \sin\left(\frac{2\pi}{n+1}\right)$. In the limit of parallel planes, this distance $= \frac{d'}{2}$. Thus $d \sin\left(\frac{2\pi}{n+1}\right) = \frac{d'}{2}$
with $d \to \infty$ and $n \to \infty \Rightarrow \frac{2d\pi}{n} = \frac{d'}{2}$ or $d = \frac{n d'}{4\pi}$.
Thus the right limits are $d = \frac{n d'}{4\pi}$ with $n \to \infty$.

2a)



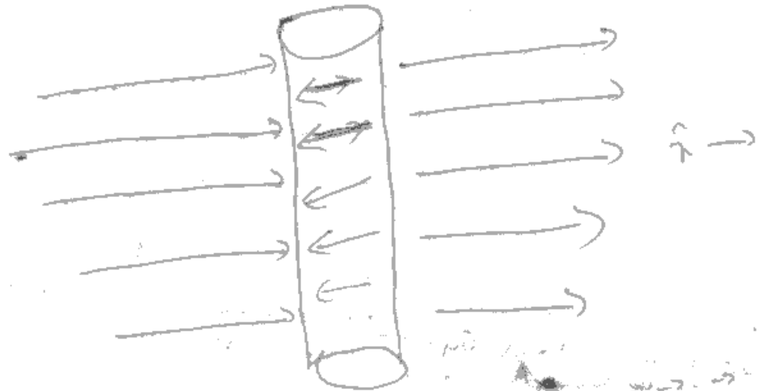
From the symmetry of the problem, it is clear that none of the quantities $\vec{E}, \vec{P}, \vec{D}, \rho_b$ and σ_b have any dependence on z.
From the azimuthal symmetry they do not have any dependence on φ as well.

~~$\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{\nabla}\times\vec{E} = 0$ inside and outside the cylinder~~

Outside the cylinder $\vec{\nabla}\cdot\vec{E} = 0$ & $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E}$ does not depend on ρ as well. Thus $\vec{E}$ is a constant outside the cylinder. Since the cylinder is not expected to have any effect on $\vec{E}$ as $\rho \to \infty$, $\vec{E}(\rho \to \infty) = E_0 \hat{z}$. Since $\vec{E}$ is a constant outside the cylinder $\Rightarrow \vec{E} = E_0 \hat{z}$ everywhere

outside the cylinder. Inside the cylinder $\vec{\nabla}\times\vec{E} = 0$ & $\vec{\nabla}\cdot\vec{D} = 0$.
Since $\vec{D} = \varepsilon\vec{E}$, $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{E}$ is a constant inside the cylinder
as well. The boundary condition on $\vec{E}$ is that $\vec{E}_{\parallel}$ is continuous.
This implies that $E_z = E_0\hat{z}$ inside the cylinder as well.
Thus $\vec{E} = E_0\hat{z}$ everywhere. This implies that $\vec{D} = \varepsilon_0 E_0\hat{z}$ outside
the cylinder and $\vec{D} = \varepsilon_0 E_0\hat{z}$ inside. $\vec{P} = 0$ outside the cylinder
and $\vec{P} = \chi E_0\hat{z} = (\varepsilon - \varepsilon_0)E_0\hat{z}$ inside the cylinder. $\rho_b = 0$
since $\vec{\nabla}\cdot\vec{P} = 0$ everywhere and $\sigma_b = 0$ since $\vec{P}$ is parallel to
the curved surface.

b) In this part, the azimuthal symmetry of the previous part is
broken. We can work with a scalar potential V such that
$\vec{E} = -\vec{\nabla}V$. The symmetry of the problem suggests that
$V(\rho, \phi)$ and not of z.



Since $\vec{\nabla}\cdot\vec{E} = 0$ outside the cylinder and $\vec{\nabla}\cdot\vec{D} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$
inside the cylinder, we have

$\nabla^2 V(\rho, \phi) = 0$ everywhere

Thus $$V(\rho, \phi) = \sum_{n=0}^{\infty}\left(A_n\rho^n + B_n\rho^{-n}\right)\left(C_n\cos n\phi + D_n\sin n\phi\right)$$

Since the region inside includes $\rho = 0$

$$V_{in}(\rho, \phi) = \sum_{n=0}^{\infty} \rho^n (\gamma_n \sin n\phi + \delta_n \cos n\phi)$$

The region ~~includes~~ outside includes $\rho = \infty$. Ordinarily this would mean that only negative powers of n have to be kept in the sum. However since the field $\vec{E}(\rho \to \infty) \neq 0$, we have to keep positive powers as well.

$$V_{out}(\rho, \phi) = \sum_{n=0}^{\infty} \rho^{-n} (\alpha_n \sin n\phi + \beta_n \cos n\phi) + \sum_{n=0}^{\infty} \rho^n (\alpha_n' \sin n\phi + \beta_n' \cos n\phi)$$

$$\vec{E} = -\vec{\nabla} V_{out} = \sum_{n=0}^{\infty} n \rho^{-(n+1)} (\alpha_n \sin n\phi + \beta_n \cos n\phi)\hat{\rho} + \sum_{n=0}^{\infty} n \rho^{-(n+1)} (\beta_n \sin n\phi - \alpha_n \cos n\phi)\hat{\phi}$$

$$- \sum_{n=0}^{\infty} n \rho^{n-1} (\alpha_n' \sin n\phi + \beta_n' \cos n\phi)\hat{\rho} + \sum_{n=0}^{\infty} n \rho^{n-1} (\beta_n' \sin n\phi - \alpha_n' \cos n\phi)\hat{\phi}$$

$$\vec{E}(\rho \to \infty, \phi) = - \sum_{n=0}^{\infty} n \rho^{n-1} (\alpha_n' \sin n\phi + \beta_n' \cos n\phi)\hat{\rho}$$

$$+ \sum_{n=0}^{\infty} n \rho^{n-1} (\beta_n' \sin n\phi - \alpha_n' \cos n\phi)\hat{\phi}$$

But $\vec{E}(\rho \to \infty, \phi) = E_0 \hat{x}$; $\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$

$\Rightarrow \alpha_n' = \beta_n' = 0 \;\; \forall n \neq 1$; $\alpha_1' = 0$; $\beta_1' = -E_0$

$$V_{out}(\rho, \phi) = -E_0 \rho \cos\phi + \sum_{n=0}^{\infty} \rho^{-n} (\alpha_n \sin n\phi + \beta_n \cos n\phi)$$

At the boundary $\rho = R$.

$V_{out}(R, \phi) = V_{in}(R, \phi) \;\; \forall \phi$

$\Rightarrow \dfrac{\alpha_n}{R^{-n}} = R^n \gamma_n \;\; \forall n$; $\dfrac{\beta_n}{R^n} = \delta_n R^n \;\; \forall n \neq 1$

$$\frac{\beta_1}{R} - E_0 R = \delta_1 R$$

The other condition is that $E_{\varphi,in}(R,\varphi) = E_{\varphi,out}(R,\varphi)\ \forall\ \varphi$

$$E_{\varphi,out}(R,\varphi) = \sum_{n=0}^{\infty} \frac{n}{R^{n+1}}\left(\beta_n \sin n\varphi - \alpha_n \cos n\varphi\right) - E_0 \sin\varphi$$

$$E_{\varphi,in}(R,\varphi) = \sum_{n=0}^{\infty} n R^{n-1}\left(\delta_n \sin\varphi - \gamma_n \cos n\varphi\right)$$

Thus, we have

$$\frac{n\beta_n}{R^{n+1}} = n R^{n-1}\delta_n \quad \forall\ n \neq 1 \quad \& \quad \frac{n\alpha_n}{R^{n+1}} = n R^{n-1}\gamma_n \quad \forall\ n$$

and $\frac{1}{R^2}\beta_1 - E_0 = \delta_1$

~~From the~~ These conditions are identical to the ones you obtain from matching the potential. The other set of boundary conditions is $D_{\rho,in} = D_{\rho,out} \Rightarrow E_{\rho,out} = \frac{\varepsilon}{\varepsilon_0} E_{\rho,in}$

$$E_{\rho,out} = E_0 \cos\varphi + \sum \frac{n}{R^{n+1}}\left(\alpha_n \sin n\varphi + \beta_n \cos n\varphi\right)$$

$$E_{\rho,in} = -\sum n R^{n-1}\left(\gamma_n \sin n\varphi + \delta_n \cos n\varphi\right)$$

Thus we obtain

$$\frac{n\alpha_n}{R^{n+1}} = -\frac{\varepsilon}{\varepsilon_0} n R^{n-1}\gamma_n \quad \forall\ n \quad \text{and} \quad \frac{n\beta_n}{R^{n+1}} = -n R^{n-1}\delta_n \frac{\varepsilon}{\varepsilon_0} \quad \forall\ n \neq 1$$

and $E_0 + \frac{\beta_1}{R^2} = -\frac{\varepsilon}{\varepsilon_0}\delta_1$

The potential and E_ρ continuity conditions give us that $\alpha_n = \gamma_n = 0\ \forall\ n$; $\beta_n = \delta_n\ \forall\ n \neq 1$ and

and $\beta_1 = E_0 R^2 \frac{(\varepsilon - \varepsilon_0)}{\varepsilon + \varepsilon_0}$ & $\delta_1 = -\frac{2\varepsilon_0 E_0}{\varepsilon + \varepsilon_0}$

Thus

$$V_{out}(\rho,\varphi) = -E_0 \rho \cos\varphi + \frac{E_0 R^2}{\rho} \frac{(\varepsilon-\varepsilon_0)}{\varepsilon+\varepsilon_0} \cos\varphi$$

$$\Rightarrow \vec{E}_{out}(\rho,\varphi) = \left[E_0 \cos\varphi + E_0 \frac{(\varepsilon-\varepsilon_0)}{\varepsilon+\varepsilon_0} \left(\frac{R}{\rho}\right)^2 \cos\varphi \right] \hat{\rho}$$

$$+ \left[\frac{E_0(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)} \left(\frac{R}{\rho}\right)^2 \sin\varphi - E_0 \sin\varphi \right] \hat{\varphi}$$

$$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}$$

$$\vec{P}_{out} = 0$$

$$V_{in}(\rho,\varphi) = -\frac{2\varepsilon_0 E_0 \rho}{\varepsilon+\varepsilon_0} \cos\varphi$$

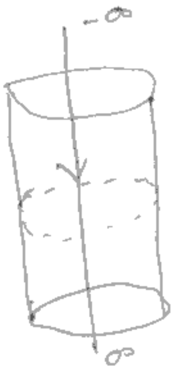
$$\vec{E}_{in}(\rho,\varphi) = \frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0} \cos\varphi\, \hat{\rho} - \frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0} \sin\varphi\, \hat{\varphi}$$

$$\vec{P}(\rho,\varphi) = (\varepsilon-\varepsilon_0)\vec{E}(\rho,\varphi)$$

$\rho_b = 0$ everywhere since $\vec{\nabla}\cdot\vec{E} = 0$ everywhere

$$\sigma_b = -\vec{P}\cdot\hat{n} = -\vec{P}\cdot\hat{\rho} = -\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}\cos\varphi$$

3a)

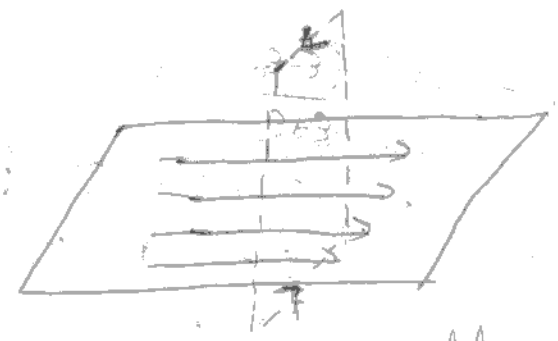


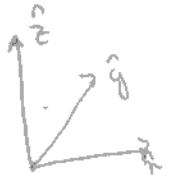
Using the right hand rule and examining the symmetry of the problem, ~~$\vec{B}$ is in the~~ it can be concluded that $\vec{B}$ is in the $-\hat{\varphi}$ direction. Using a circular Amperian loop

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I \quad \Rightarrow -B_\varphi 2\pi\rho = \mu_0 I$$

$$\text{so } B_\varphi = -\frac{\mu_0 I}{2\pi\rho}; \quad \vec{B} = -\frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$$

b)





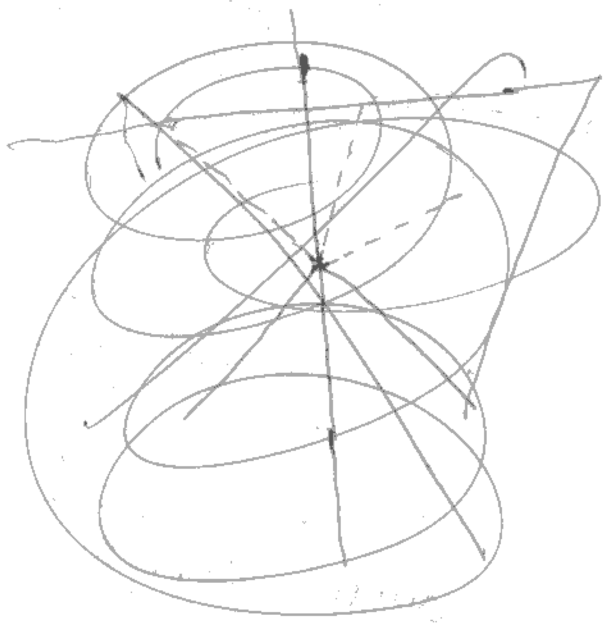
From the symmetry of the problem and the right hand rule, and independent of x and y it can be seen that $\vec{B}$ is in the $\hat{y}$ direction. Further $\vec{B}(z) = -\vec{B}(-z)$. Considering a rectangular Amperian loop in the yz plane going from +z to −z of side length Δl,

we obtain ~~$B_z \Delta l$~~ $[B(z) - B(-z)]\Delta l = -\mu_0 K \Delta l$ $\Rightarrow$ $B(z) = -\frac{\mu_0 K}{2}$ and $B(-z) = \frac{\mu_0 K}{2}$

or $B(z) - B(-z) = -\mu_0 K$

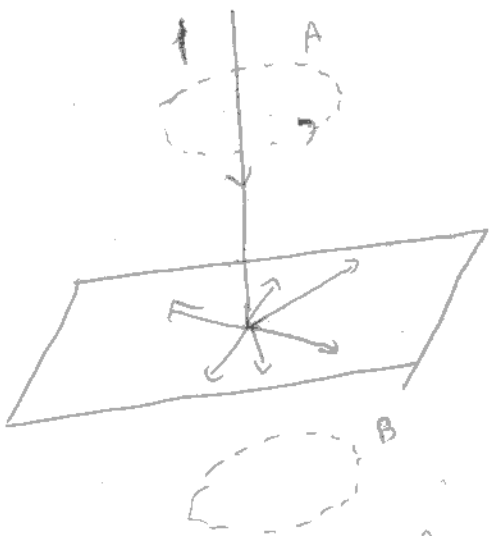
So $\vec{B}(x,y,z) = -\frac{\mu_0 K}{2}\hat{y}$ $\quad z > 0$

$= \frac{\mu_0 K}{2}\hat{y}$ $\quad z < 0$



1(c)

From the symmetry of the problem, it can be argued that $\vec{B}$ is independent of φ. Further it can also be argued that $\vec{B}$ is only along $\hat{\varphi}$. This can be seen using the right hand rule for the wire along the z axis and each filament of current coming out from the origin. ~~Further, the azimuthal sy~~



The Amperian loop A above the plane thus gives

$\oint \vec{B}\cdot d\vec{l} = -\mu_0 I$ or $B_\varphi(\rho, z)\, 2\pi\rho = -\mu_0 I$

$\Rightarrow \vec{B}(\rho, \varphi, z>0) = -\dfrac{\mu_0 I}{2\pi\rho}\hat{\varphi}$

The Amperian loop B below the plane gives

$\oint \vec{B}\cdot d\vec{l} = 0 \Rightarrow \vec{B}(\rho, \varphi, z<0) = 0$