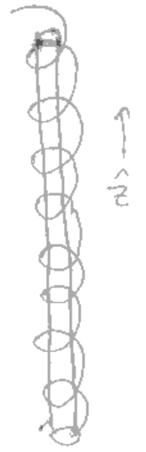
1)




a) The symmetry of the problem suggests that $\vec{B}$, $\vec{H}$ and $\vec{m}$ are all along $\hat{z}$.

$$\oint_C \vec{H} \cdot d\vec{l} = n I \; ;$$

where C is a standard Amperian loop with one side inside the solenoid and parallel to its axis and a parallel side at $\infty$. The other two sides are perpendicular to these two sides; and since $\vec{H} \parallel \hat{z}$, we obtain $\vec{H}(\infty) = 0$ and

$$\vec{H} = \frac{n I}{L} \hat{z} = N I \hat{z} \quad \text{everywhere inside the solenoid}$$

Here L is the length of the side inside the solenoid and n is the # of turns enclosed by the loop.

Using a similar loop that is completely outside the solenoid, we see that $\vec{H} = 0$ everywhere outside. Thus the fields are

$\vec{H} = N I \hat{z}$ inside the solenoid and zero outside

$\vec{B} = \mu N I \hat{z}$ inside the solenoid and zero outside

$\vec{m} = \left(\frac{\mu}{\mu_0} - 1\right) N I \hat{z}$ inside the solenoid and zero outside

b) We assume that the fields are along $\hat{z}$ and show that we can find values consistent with the boundary conditions.

Once again we consider an Amperian loop with one side parallel to the axis of the solenoid and inside the region containing the magnetic material. The other parallel side is at $\infty$ and the remaining two sides are perpendicular to the first two sides. As in 1 a) we obtain

$\vec{H} = NI\,\hat{z}$ for $a < r \leq R$ and zero outside $(r > R)$

Since there are no free currents inside the solenoid,

$\vec{H} = NI\,\hat{z}$ for $r < a$ as well.

Thus, the fields are.

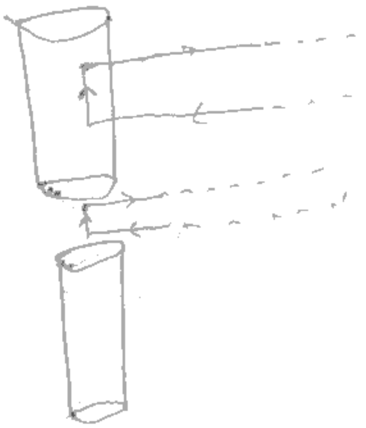
$\vec{H} = NI\,\hat{z}$ for $r < R$ and zero for $r > R$

$\vec{B} = \mu_0 NI\,\hat{z}$ for $r < a$, $\mu NI\,\hat{z}$ for $a < r < R$, zero for $r > R$

$\vec{m} = 0$ for $r < a$, $\left(\frac{\mu}{\mu_0} - 1\right) NI\,\hat{z}$ for $a < r < R$, zero for $r > R$.

These fields satisfy all the required boundary conditions and hence all the fields are along $\hat{z}$.

c) In this case, consider an Amperian loop whose side parallel to the axis of the solenoid is contained completely inside one of the halves of the magnetic material as shown below:



Consider another Amperian loop with the parallel side completely inside the gap also as shown in the figure. We obtain $\vec{H} = NI\,\hat{z}$ for both inside the magnetic material and in the gap from arguments similar to those used in 1a) & 1b).

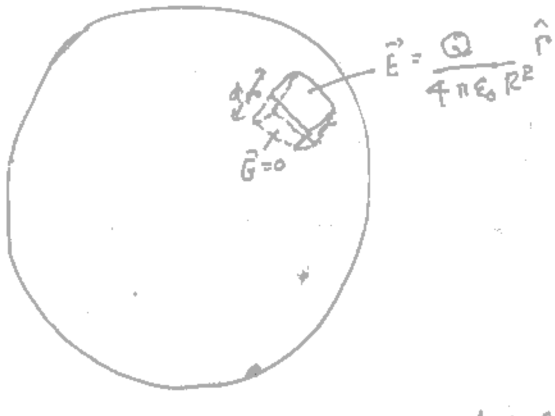
Thus, we obtain for the magnetic field

$\vec{B} = \mu NI\,\hat{z}$ inside the magnetic material

$\vec{B} = \mu_0 NI\,\hat{z}$ inside the gap

which implies that $B_\perp$ is not continuous across the gap-material interface violating the boundary condition for $B_\perp$. Thus, the fields <u>cannot</u> be along $\hat{z}$.

2)




a) Recall that the force on a volume bounded by a surface S is $F_\alpha = \oiint_S T_{\alpha\beta}\, dS_\beta$. Here we consider a small volume with the patch as the outside surface and thickness $dr$. The electric field is zero ~~at~~ all the surfaces except the outside one. Thus

$$F_\alpha = \iint_{\text{outside}} T_{\alpha\beta}\, dS_\beta$$

~~On this S~~

On the outside surface, $d\vec{S} = dA\,\hat{r}$ and $\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{r}$

Thus $d\vec{S} = dA\,\frac{4\pi\epsilon_0 R^2}{Q}\vec{E} = dA\,\frac{\vec{E}}{|\vec{E}|}$

$$T_{\alpha\beta}\,dS_\beta = \epsilon_0\left(E_\alpha E_\beta - \frac{1}{2}\delta_{\alpha\beta}|\vec{E}|^2\right)\frac{E_\beta}{|\vec{E}|}\,dA$$

$$= \epsilon_0\left(E_\alpha|\vec{E}|^2 - \frac{1}{2}E_\alpha|\vec{E}|^2\right)\frac{dA}{|\vec{E}|}$$

$$= \frac{1}{2}\epsilon_0 E_\alpha|\vec{E}|\,dA$$

$$\Rightarrow \text{Force} = \frac{1}{2}\epsilon_0\frac{Q^2}{16\pi^2\epsilon_0^2 R^4}\hat{r}\,dA$$

From electrostatics, we know that

$$\vec{E}_{out} = \vec{E}_{rest} + \vec{E}_{patch} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{r}$$

$$\vec{E}_{in} = \vec{E}_{rest} - \vec{E}_{patch} = 0$$

Here $\vec{E}_{in}$ and $\vec{E}_{out}$ are the electric fields just inside and outside the surface. $\vec{E}_{patch}$ is the electric field just outside the surface due to the patch and $\vec{E}_{rest}$ is the field due to the rest of the sphere, which is continuous across the patch. Thus

$$\vec{E}_{rest} = \frac{1}{2}\frac{Q}{4\pi\epsilon_0 R^2}\hat{r}$$

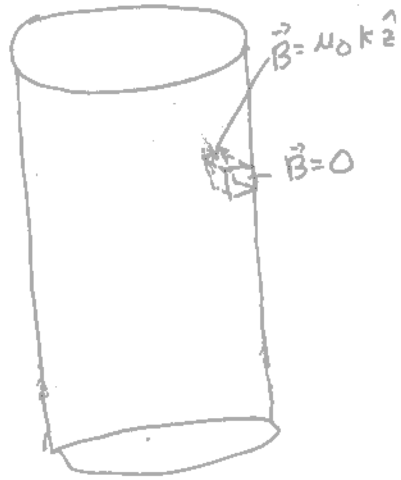
The force on the patch is

$$d\vec{F} = dq\,\vec{E}_{rest} = \frac{Q}{4\pi R^2}dA\left(\frac{1}{2}\frac{Q}{4\pi\epsilon_0 R^2}\right)\hat{r} = \frac{1}{2}\epsilon_0\frac{Q^2}{16\pi^2\epsilon_0^2 R^4}\hat{r}$$

as before. Here $dq$ is the charge on the patch given by

$$dq = \frac{Q}{4\pi R^2}dA$$

2 b)




Consider a volume whose outside surface is just outside the cylinder above the ~~part~~ patch and the inner surface is just inside the cylinder below the patch. The ~~f~~ magnetic field is $\vec{B} = \mu_0 K \hat{z}$ everywhere in this volume except on the outer surface where it is zero. The components of the stress tensor every where inside, are thus

$T_{zz} = \frac{1}{2} \mu_0 K^2$, $T_{xx} = T_{yy} = -\frac{1}{2} \mu_0 K^2$; $T_{xy} = T_{yx} = T_{yz} = T_{zy} = T_{zx} = T_{xz} = 0$

Once again

$$F_\alpha = \oiint T_{\alpha\beta} \, dS_\beta$$

~~Here~~ Now

$$F_\alpha = \int_{\text{inner}} T_{\alpha\beta} \, dS_\beta + \int_{\text{top}} T_{\alpha\beta} \, dS_\beta + \int_{\text{bottom}} T_{\alpha\beta} \, dS_\beta$$

since $T_{\alpha\beta} = 0$ on the outer surface.

It is easy to see that $\int_{\text{top}} T_{\alpha\beta} \, dS_\beta = -\int_{\text{bottom}} T_{\alpha\beta} \, dS_\beta$

Thus $F_\alpha = \int_{\text{inner}} T_{\alpha\beta} \, dS_\beta$

$d\vec{S} = -dA\,\hat{r}$ in cylindrical polar coordinates

$= \cancel{-dA\,(x\hat{x}+y\hat{y})}$,

$$d\vec{S} = -dA\,\frac{x\hat{x}+y\hat{y}}{R}$$

$$\text{Force} = \int_{\text{inner}} T_{\alpha\beta}\,dS_\beta = dA\,\frac{1}{2}\mu_0 K^2\left(\frac{x\hat{x}+y\hat{y}}{R}\right) = \frac{1}{2}\mu_0 K^2\,dA\,\hat{r}$$

Let us calculate the force from magnetostatics

Like in 2a)

$$\vec{B}_{in} = \vec{B}_{rest} - \vec{B}_{patch} = \mu_0 K\hat{z}$$

$$\vec{B}_{out} = \vec{B}_{rest} + \vec{B}_{patch} = 0$$

$$\Rightarrow \vec{B}_{rest} = \frac{\mu_0 K\hat{z}}{2}$$

The force

$$d\vec{F} = \vec{K}\times\vec{B}_{rest}\,dA$$
$$= K\hat{\varphi}\times\frac{1}{2}\mu_0 K\hat{z}\,dA = \frac{1}{2}\mu_0 K^2\,dA\,\hat{r}$$ as calculated above.

3a) The expression for the momentum of $\vec{E}$ & $\vec{B}$ fields is

$$\vec{P} = \varepsilon_0\int(\vec{E}\times\vec{B})\,d^3\vec{r}$$

$\vec{E} = -\vec{\nabla}\varphi$ so

$$\vec{P} = \varepsilon_0\int(-\vec{\nabla}\varphi\times\vec{B})\,d^3\vec{r} = \varepsilon_0\int\varphi\,\vec{\nabla}\times\vec{B}\,d^3\vec{r} - \varepsilon_0\int\vec{\nabla}\times(\varphi\vec{B})\,d^3\vec{r}$$

using the identity $\vec{\nabla}\times(\varphi\vec{B}) = \varphi\,\vec{\nabla}\times\vec{B} + \vec{\nabla}\varphi\times\vec{B}$

Consider the integral

$$\vec{I} = \int \vec{\nabla}\times(\varphi\vec{B})\, d^3\vec{r}$$

Let $\vec{C}$ be a constant vector

$$\vec{C}\cdot\vec{I} = \int \vec{C}\cdot\vec{\nabla}\times(\varphi\vec{B})\, d^3\vec{r} = \int \vec{\nabla}\cdot(\varphi\vec{B}\times\vec{C})\, d^3\vec{r}$$

$$= \int_{\text{infinity}} (\varphi\vec{B}\times\vec{C})\cdot d\vec{S} = -\int_{\infty} (\vec{C}\times\varphi\vec{B})\cdot d\vec{S} = \vec{C}\cdot\int_{\infty} d\vec{S}\times\varphi\vec{B}$$

where we have used the divergence theorem to convert the volume integral into a surface integral. Since $\vec{C}$ is any constant vector

$$\int \vec{\nabla}\times(\varphi\vec{B})\, d^3\vec{r} = \int_{\infty} d\vec{S}\times\varphi\vec{B} \to 0 \text{ if } \varphi\vec{B} \text{ falls off faster than } \frac{1}{r^2}.$$

Thus

$$\vec{p} = \varepsilon_0 \int \varphi\,\vec{\nabla}\times\vec{B}\, d^3\vec{r} = \mu_0\varepsilon_0 \int \varphi\,\vec{J}\, d^3\vec{r} = \frac{1}{c^2}\int \varphi\,\vec{J}\, d^3\vec{r}$$

b) Let us take the origin to be somewhere inside the region where the current distribution exists. Since, $\vec{E} = -\vec{\nabla}\varphi$ varies slowly over the spatial extent of $\vec{J}$, we can write $\varphi(\vec{r}) = \varphi(0) - \vec{r}\cdot\vec{E}$ with $\vec{E}$ constant. Thus

$$\vec{p} = \frac{1}{c^2}\int (\varphi(0) - \vec{r}\cdot\vec{E})\,\vec{J}\, d^3\vec{r} = \frac{\varphi(0)}{c^2}\int \vec{J}\, d^3\vec{r} - \frac{1}{c^2}\int (\vec{r}\cdot\vec{E})\,\vec{J}\, d^3\vec{r}$$

The first integral is zero since $\int \vec{J}\, d^3\vec{r} = 0$

The second integral is

$$\int (\vec{r}\cdot\vec{E})\,\vec{J}\,d^3\vec{r} = E_i \int x_i J_j\,d^3\vec{r} = -E_i \int x_j J_i\,d^3\vec{r}$$

from the ~~identity~~ given identity. Thus

$$\int (\vec{r}\cdot\vec{E})\,\vec{J}\,d^3\vec{r} = -\int (\vec{J}\cdot\vec{E})\,\vec{r}\,d^3\vec{r}$$

$\Rightarrow$

$$2\int (\vec{r}\cdot\vec{E})\,\vec{J}\,d^3\vec{r} = \int (\vec{r}\cdot\vec{E})\,\vec{J}\,d^3\vec{r} - \int (\vec{J}\cdot\vec{E})\,\vec{r}\,d^3\vec{r}$$

$$= \vec{E}\times\int (\vec{J}\times\vec{r})\,d^3\vec{r}$$

or $$\int (\vec{r}\cdot\vec{E})\,\vec{J}\,d^3\vec{r} = -\vec{E}\times\vec{m}$$

since the magnetic moment

$$\vec{m} = \frac{1}{2}\int (\vec{r}\times\vec{J})\,d^3\vec{r}$$

Thus $\vec{P} = \frac{1}{c^2}\vec{E}\times\vec{m}$