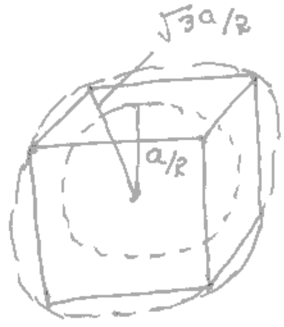
EXAM I

SOLUTIONS

1)



Consider the two spheres in the figure above. The inner sphere S_1 touching the faces of the cube has radius $r_1 = \frac{a}{2}$ (i.e. half the distance between parallel faces).

The capacitance of a sphere of radius R is $C = 4\pi\varepsilon_0 R$.

Thus $C_{S_1} = 4\pi\varepsilon_0\left(\frac{a}{2}\right) = 2\pi\varepsilon_0 a.$

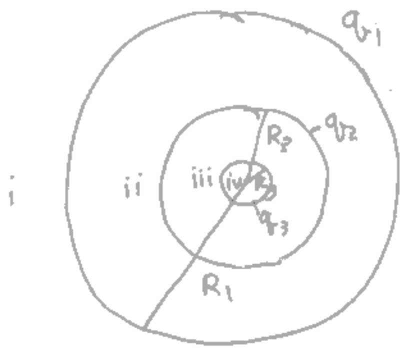
The outer sphere S_2 touching the ~~edges~~ vertices of the cube, which contains the cube has radius $r_2 = \frac{\sqrt{3}a}{2}$ (i.e. half the distance between vertices connected by a body diagonal).

Thus $C_{S_2} = 4\pi\varepsilon_0\left(\frac{\sqrt{3}a}{2}\right) = 2\sqrt{3}\pi\varepsilon_0 a$

Since the volume of the cube is inbetween that of S_1 and S_2,

$$2\pi\varepsilon_0 a < C_{cube} < 2\sqrt{3}\pi\varepsilon_0 a$$

2)



Let us first calculate the electric fields. From the spherical symmetry of the problem, $\vec{E}$ is only ~~a~~ function of r, the distance from the centre and directed radially outwards. Further, it only depends on the total charge enclosed in a sphere of radius r.

Thus,

in i) $\vec{E}_i(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(q_1+q_2+q_3)}{r^2}\hat{r}$

in ii) $E_{ii}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(q_2+q_3)}{r^2}\hat{r}$

in iii) $E_{iii}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r^2}\hat{r}$

in iv) $E_{iv}(\vec{r}) = 0$

a) The potential $V(r) = -\int_{\infty}^{r}\vec{E}\cdot d\vec{r}$

In i) $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{(q_1+q_2+q_3)}{r} = -\int_{\infty}^{r}\vec{E}_i\cdot d\vec{r}$

In ii) $V(r) = -\frac{1}{4\pi\varepsilon_0}\left[\int_{-\infty}^{R_1}\vec{E}_i\cdot d\vec{r} + \int_{R_1}^{r}\vec{E}_{ii}\cdot d\vec{r}\right]$

$= \frac{1}{4\pi\varepsilon_0}\left[\frac{(q_1+q_2+q_3)}{R_1} + (q_2+q_3)\left(\frac{1}{r}-\frac{1}{R_1}\right)\right]$

$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{R_1} + \frac{(q_2+q_3)}{r}\right]$

In iii) $V(r) = -\frac{1}{4\pi\epsilon_0}\left[\int_\infty^{R_1} \vec{E}_i \cdot d\vec{r} + \int_{R_1}^{R_2} \vec{E}_{ii} \cdot d\vec{r} + \int_{R_2}^{r} \vec{E}_{iii} \cdot d\vec{r}\right]$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1+q_2+q_3}{R_1} + (q_2+q_3)\left(\frac{1}{R_2}-\frac{1}{R_1}\right) + q_3\left(\frac{1}{r}-\frac{1}{R_2}\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{R_1} + \frac{q_2}{R_2} + \frac{q_3}{r}\right]$$

In iv) $V(r) = V(R_3)$ since $E_{iv} = 0$

Thus $V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{R} + \frac{q_2}{R_2} + \frac{q_3}{R_3}\right]$

b) Energy $= \frac{1}{2}\left[q_1 V(R_1) + q_2 V(R_2) + q_3 V(R_3)\right]$

$$= \frac{1}{8\pi\epsilon_0}\left[\frac{q_1(q_1+q_2+q_3)}{R_1} + \frac{q_2 q_1}{R_1} + \frac{q_2(q_2+q_3)}{R_2} + \frac{q_3 q_1}{R_1} + \frac{q_3 q_2}{R_2} + \frac{q_3^2}{R_3}\right]$$

$$= \frac{1}{8\pi\epsilon_0}\left[\frac{q_1^2}{R_1} + \frac{q_2^2}{R_2} + \frac{q_3^2}{R_3} + \frac{2q_1q_2}{R_1} + \frac{2q_1q_3}{R_1} + \frac{2q_2q_3}{R_2}\right]$$

c) When the innermost and outermost spheres are connected, their potentials are equal, $V(R_1) = V(R_3)$. Further, if the new charges on them are q_1' and q_3', $q_1' + q_3' = q_1 + q_3$ from charge conservation. The charge on the middle sphere q_2 remains unchanged.

Thus $\frac{1}{4\pi\epsilon_0}\frac{(q_1' + q_2 + q_3')}{R_1} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1'}{R_1} + \frac{q_2}{R_2} + \frac{q_3'}{R_3}\right)$

or $q_3' = \frac{q_2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}{\frac{1}{R_3} - \frac{1}{R_1}} = q_2 \frac{R_3}{R_2}\frac{(R_2 - R_1)}{R_1 - R_3}$

$$q_1' = q_1 + q_3 - q_3' = q_1 + q_3 - q_R \frac{R_3}{R_2}\frac{(R_2-R_1)}{R_1-R_3}$$

3a) $$V(x,y) = \sum_{k\neq 0} \alpha_k e^{kx}\cos ky + \beta_k e^{-kx}\cos ky + \gamma_k e^{kx}\sin ky + \delta_k e^{-kx}\sin ky)$$
$$+ \epsilon xy + \eta x + \rho y + \xi$$

$$\cancel{V(0,y) = \sum_{k\neq 0}(\alpha_k \cos ky + \beta_k \cos ky) + \rho y + \xi = 0 \Rightarrow \alpha_k, \beta_k, \rho, \xi = 0}$$

$$V(x,0) = \sum_{k\neq 0}(\alpha_k e^{kx} + \beta_k e^{-kx}) + \eta x + \xi = 0 \Rightarrow \alpha_k, \beta_k, \eta, \xi = 0$$

$$V(x,a) = \sum_{k\neq 0}(\gamma_k e^{kx}\sin ka + \delta_k e^{-kx}\sin ka) + \epsilon a x + \rho a = 0$$

$\Rightarrow \epsilon, \rho = 0$ and $\sin ka = 0$ or $k = \frac{n\pi}{a}$, n integer

$$V(x,y) = \sum_{n=1}^{\infty}\left[\gamma_n e^{\frac{n\pi x}{a}}\sin\frac{n\pi y}{a} + \delta_n e^{-\frac{n\pi x}{a}}\sin\frac{n\pi y}{a}\right]$$

$$V(0,y) = \sum_{n=1}^{\infty}(\gamma_n + \delta_n)\sin\frac{n\pi y}{a} = 0 \Rightarrow \gamma_n = -\delta_n$$

$$V(x,y) = \sum_{n=1}^{\infty} 2\gamma_n \sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a}$$

$$V(a,y) = V_0 \Rightarrow \sum_{n=1}^{\infty} 2\gamma_n \sinh n\pi \sin\frac{n\pi y}{a} = V_0$$

or $$2\gamma_n \sinh n\pi \frac{a}{2} = V_0\int_0^a \sin\frac{n\pi y}{a}\,dy = \frac{2V_0 a}{n\pi} \quad (n \text{ odd})$$
$$= 0 \quad (n \text{ even})$$

using $$\int_0^a \sin\frac{n\pi y}{a}\sin\frac{m\pi y}{a}\,dy = \frac{a}{2}\delta_{mn}$$

$$\Rightarrow 2\gamma_n = \frac{4V_0}{n\pi \sinh n\pi} \quad (n \text{ odd})$$
$$= 0 \quad (n \text{ even})$$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\frac{\sinh\frac{n\pi x}{a}\sin\left(\frac{n\pi y}{a}\right)}{\sinh n\pi}$$

$\vec{E} = -\vec{\nabla} V$

$$\vec{E} = -\frac{4V_0}{a} \sum_{n=odd} \frac{1}{\sinh n\pi} \left[\cosh\frac{n\pi x}{a} \sin\frac{n\pi y}{a}\, \hat{x} + \sinh\frac{n\pi x}{a} \cos\frac{n\pi y}{a}\, \hat{y} \right]$$

3 b) The potential $V'(x,y)$ in this case is simply

$$V'(x,y) = V_0 - V(x,y)$$

It can be easily seen that $\nabla^2 V' = 0$ and it satisfies $V'(x,0) = V'(x,a) = V'(0,y) = V_0$ and $V'(a,y) = 0$, so by the uniqueness theorem, it is the right solution.