PH 206

FINAL EXAM: SOLUTIONS

1) $J_\alpha = \sigma_{\alpha\beta} E_\beta$

Let us assume that the general state of polarization of a beam can be represented by a column vector.

$$E = \begin{pmatrix} E_x \\ E_y \end{pmatrix} \quad . \quad E_z = 0 \text{ since } \vec{k} \parallel \hat{z}$$

Right ~~Left~~ circulary polarized (RCP) light is of the form

$$E = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Left ~~Right~~ circularly polarized (LCP) light is of the form

$$E = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

a) Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\overleftrightarrow{\sigma} \vec{E} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

For waves with $\vec{k} \parallel \hat{z}$, we obtain

$$k^2 E_\alpha = -i\mu_0 \omega \sigma_{\alpha\beta} E_\beta + \frac{\omega^2}{c^2} E_\alpha$$

(where $\alpha = x, y$)

This is a matrix equation of the form

$$k^2 \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{\omega^2}{c^2} & -i\mu_0\omega\sigma_H \\ +i\mu_0\omega\sigma_H & \frac{\omega^2}{c^2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

The matrix $M = \begin{pmatrix} \frac{\omega^2}{c^2} & i\mu_0\omega\sigma_H \\ -i\mu_0\omega\sigma_H & \frac{\omega^2}{c^2} \end{pmatrix} = \frac{\omega^2}{c^2}\mathbb{I} + \mu_0\omega\sigma_H S_y$

where $\mathbb{I}$ is the identity matrix and $S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

is the second Pauli spin matrix

It can easily be verified that the the vectors for RCP and LCP are eigenvectors of this matrix and thus have definite ω vs. k relations.

b) Eigenvalue corresponding to RCP is

$$k^2 \begin{pmatrix} 1 \\ +i \end{pmatrix} = \begin{pmatrix} \frac{\omega^2}{c^2} & -i\mu_0\omega\sigma_H \\ +i\mu_0\omega\sigma_H & \frac{\omega^2}{c^2} \end{pmatrix} \begin{pmatrix} 1 \\ +i \end{pmatrix} = \frac{\omega^2}{c^2} + \mu_0\omega\sigma_H \begin{pmatrix} 1 \\ +i \end{pmatrix}.$$

Thus the ~~eigen~~ eigenvalue is $k = \sqrt{\omega\left(\frac{\omega}{c^2} + \mu_0\sigma_H\right)}$

Similarly eigenvalue for LCP is

$$k^2 \begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix}\frac{\omega^2}{c^2} & -i\mu_0\omega\sigma_H \\ +i\mu_0\omega\sigma_H & \frac{\omega^2}{c^2}\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\omega^2}{c^2} - \mu_0\omega\sigma_H \begin{pmatrix}1\\-i\end{pmatrix}$$

Thus, the eigenvalue is $k = \sqrt{\omega\left(\frac{\omega}{c^2} - \mu_0\sigma_H\right)}$

Thus, there is a critical frequency $\omega_c = \mu_0\sigma_H c^2$ below which LCP will not propagate.

A simple way to get both the eigenvalues and eigenvectors is to use the fact that

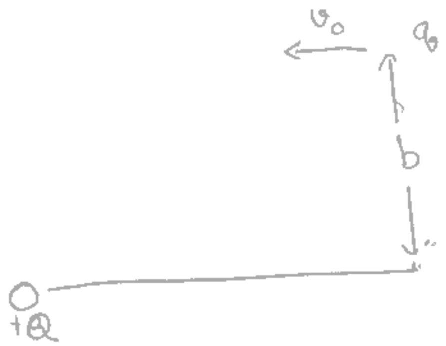
$$M = \frac{\omega^2}{c^2}\mathbb{I} + \mu_0\omega\sigma_H S_y$$

Thus, the eigenvectors of M are simply those of S_y which are $\begin{pmatrix}1\\i\end{pmatrix}$ and $\begin{pmatrix}1\\-i\end{pmatrix}$ corresponding to RCP and LCP respectively. Their eigenvalues with S_y are $+1$ and -1 respectively, giving

$$k = \sqrt{\omega\left(\frac{\omega}{c^2} + \mu_0\sigma_H\right)} \quad \text{RCP}$$

and

$$k = \sqrt{\omega\left(\frac{\omega}{c^2} - \mu_0\sigma_H\right)} \quad \text{LCP}$$



a) If Q did not exist, q would not feel any EM forces and would ~~contiue~~ move along a straight line with speed v_0. ~~The~~ Further its motion would be non-relativistic if $v_0 << c$. If we want the motion of the particle to essentially be along a straight line and non-relativistic, in addition to $v_0 << c$, we need the strength of EM interactions to be much less than the particle's kinetic energy. For a non-relativistic particle, the EM interaction is just the Coulomb interaction, whose natural scale in this problem is $\frac{1}{4\pi\varepsilon_0}\frac{Qq}{b}$. Thus, we need

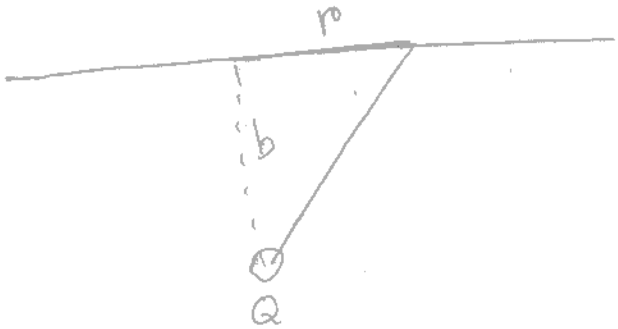
$$\frac{1}{4\pi\varepsilon_0}\frac{Qq}{b} << \frac{1}{2} m v_0^2 \quad \text{or} \quad v_0 >> \sqrt{\frac{Qq}{2\pi\varepsilon_0 m b}}$$

in addition to $v_0 << c$. Combining both requirements, we have

$$\sqrt{\frac{Qq}{2\pi\varepsilon_0 m b}} << v_0 << c.$$

b) Since, the motion is non-relativistic, we have

$$m|\vec{a}| = \frac{1}{4\pi\varepsilon_0} \frac{Qq}{(b^2+r^2)}$$



Here, we have also used the fact the motion is along a straight line so that the ~~distance~~ perpendicular distance to Q is always b for the moving charge.

Thus
$$|\vec{a}| = \frac{1}{4\pi\varepsilon_0 m} \frac{Qq}{b^2+r^2}$$

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2}{3} \frac{q^2 a^2}{c^3} = \frac{2}{3c^3} \frac{1}{(4\pi\varepsilon_0)^3} \frac{Q^2 q^4}{m^2 (b^2+r^2)^2}$$

$$P = \frac{dE}{dt}$$

Thus $dE = \dfrac{2}{3c^3} \dfrac{1}{(4\pi\varepsilon_0)^3} \dfrac{Q^2 q^4}{m^2} \dfrac{dt}{(b^2+r^2)^2}$

Since the motion is essentially along a straight line.

$dt \approx \dfrac{dr}{v_0}$ (There is no perceptibly change in the ~~...~~ speed in the direction of motion)

$$E = \frac{2}{3c^3} \frac{1}{(4\pi\varepsilon_0)^3} \frac{Q^2 q^4}{m^2 v_0} \int_{-\infty}^{\infty} \frac{dr}{(b^2+r^2)^2}$$

$$= \frac{2}{3c^3} \frac{1}{(4\pi\varepsilon_0)^3} \frac{Q^2 q^4}{m^2 v_0 b^3} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{3c^3} \frac{1}{(4\pi\varepsilon_0)^3} \frac{Q^2 q^4}{m^2 v_0 b^3}$$

(Here $x = \frac{r}{b}$)

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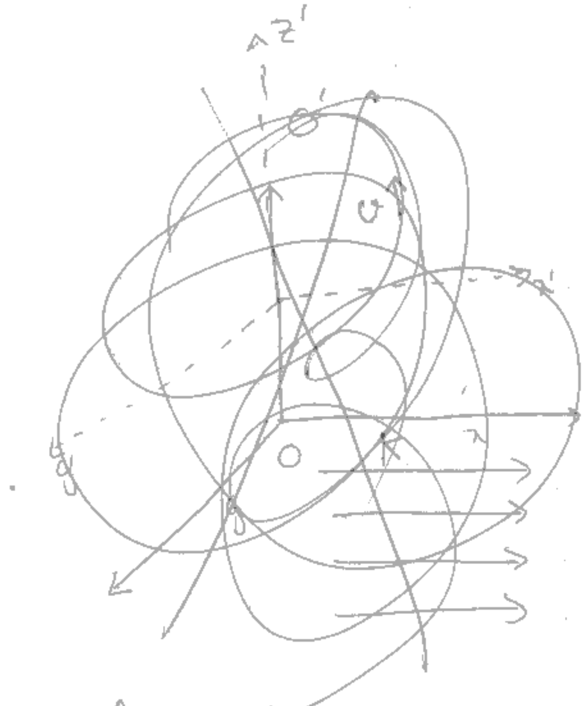

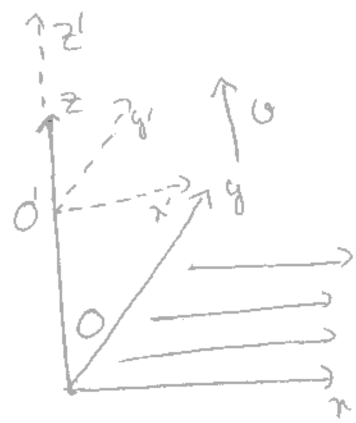

a) $\vec{E} = 0$ everywhere in O

$$\vec{B} = -\frac{\mu_0 K}{2}\, \mathrm{Sgn}(z)\, \hat{y}$$

b) Let us call

$$\vec{B} = -\frac{\mu_0 K}{2}\, \mathrm{Sgn}(z)\, \hat{y} = B(z)\, \hat{y}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B(z) \\ 0 & 0 & 0 & 0 \\ 0 & -B(z) & 0 & 0 \end{pmatrix}$$

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta}$$

The only non-zero values of $F^{\alpha\beta}$ are for $(\alpha\, \beta)$ ~~are~~ (13) and (31)

$$F'^{\mu\nu} = \left(\frac{\partial x'^{\mu}}{\partial x^{1}} \frac{\partial x'^{\nu}}{\partial x^{3}} - \frac{\partial x'^{\mu}}{\partial x^{3}} \frac{\partial x'^{\nu}}{\partial x^{1}} \right) B(z)$$

Thus, we only need to calculate $\frac{\partial x'^{\mu}}{\partial x^1}$ and $\frac{\partial x'^{\mu}}{\partial x^3}$

- For the Lorentz transformations from O to O', only the following derivatives ~~are nonzero~~ that we require are non-zero

$$\frac{\partial x'^1}{\partial x^1} = 1 \; ; \; \frac{\partial x'^3}{\partial x^3} = \gamma \; ; \; \frac{\partial x'^0}{\partial x^3} = -\gamma \frac{v}{c}$$

Thus, the only non-zero components of $F'^{\mu\nu}$ are

$$F'^{10} = -F'^{01} = -\frac{\gamma v}{c} B(z)$$

$$F'^{13} = -F'^{31} = \gamma B(z)$$

Thus

$$F'^{\mu\nu} = \begin{pmatrix} 0 & \frac{\gamma v}{c} B(z) & 0 & 0 \\ -\frac{\gamma v}{c} B(z) & 0 & 0 & \gamma B(z) \\ 0 & 0 & 0 & 0 \\ 0 & -\gamma B(z) & 0 & 0 \end{pmatrix}$$

This is $F'^{\mu\nu}$ expressed in terms of $\vec{r}$ and t. To express it in terms of $\vec{r}'$ and t', we note that $\mathrm{Sgn}(z) = \mathrm{Sgn}(z'+vt')$

Thus

$$\vec{E}'(\vec{r}', t') = \gamma v \frac{\mu_0 K}{2} \mathrm{Sgn}(z'+vt') \hat{y}$$

$$\vec{B}(\vec{r}', t') = -\gamma \frac{\mu_0 K}{2} \mathrm{Sgn}(z'+vt') \hat{y}$$

The four current transforms as a Lorentz vector

In O,

$$J^0 = 0 \; ; \; J^1 = K\delta(z) \; ; \; J^2 = 0 \; ; \; J^3 = 0$$

This implies that

$$J'^1 = J^1 = K\delta(z) = K\delta[\gamma(z'+vt')] = \frac{K}{\gamma}\delta(z'+vt')$$

$$J'^0 = J'^2 = J'^3 = 0$$

Thus $\vec{J}'(\vec{r}',t') = \frac{K}{\gamma}\delta(z'+vt')\,\hat{x}'$

$$\rho'(\vec{r}',t') = 0$$

c) $\vec{E}'(\vec{r}',t') = \gamma v \frac{\mu_0 K}{2} \operatorname{Sgn}(z'+vt')\,\hat{x}'$

$$\vec{B}'(\vec{r}',t') = -\gamma \frac{\mu_0 K}{2} \operatorname{Sgn}(z'+vt')\,\hat{y}'$$

$$\vec{J}'(\vec{r}',t') = \frac{K}{\gamma}\delta(z'+vt')\,\hat{x}'$$

$$\rho'(\vec{r}',t') = 0$$

$\vec{\nabla}'\cdot\vec{E}' = 0$ & $\vec{\nabla}'\cdot\vec{B}' = 0$ can be deduced very easily.

$$\vec{\nabla}'\times\vec{E}' = \frac{\gamma v \mu_0 K}{2}\cdot\frac{\partial}{\partial z'}\operatorname{Sgn}(z'+vt')\,\hat{y}' = -\gamma v \mu_0 K\,\delta(z'+vt')\,\hat{y}'$$

$$-\frac{\partial \vec{B}'}{\partial t'} = \frac{\gamma \mu_0 K}{2}\,\frac{\partial \operatorname{Sgn}(z'+vt')}{\partial t'}\,\hat{y}' = -\gamma v \mu_0 K\,\delta(z'+vt')\,\hat{y}'$$

Thus $\vec{\nabla}'\times\vec{E}' = -\frac{\partial \vec{B}'}{\partial t'}$

Finally

$$\vec{\nabla}'\times\vec{B}' = \frac{\gamma\mu_0 K}{2}\,\frac{\partial \operatorname{Sgn}(z'+vt')}{\partial z'}\,\hat{x}' = \gamma\mu_0 K\,\delta(z'+vt')\,\hat{x}'$$

$$\frac{1}{c^2}\frac{\partial \vec{E}'}{\partial t'} = \frac{\gamma v \mu_0 K}{2c^2}\,\frac{\partial \operatorname{Sgn}(z'+vt')}{\partial t'}\,\hat{x}' = \frac{\gamma v^2}{c^2}\mu_0 K\,\delta(z'+vt')\,\hat{x}'$$

$$\mu_0\vec{J}' + \frac{1}{c^2}\frac{\partial \vec{E}'}{\partial t'} = \mu_0 K\,\delta(z'+vt')\,\hat{x}'\left(\frac{1}{\gamma} + \frac{\gamma v^2}{c^2}\right) = \gamma\mu_0 K\,\delta(z'+vt')\,\hat{x}'\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right)$$

$$= \gamma\mu_0 K\,\delta(z'+vt')\,\hat{x}' = \vec{\nabla}'\times\vec{B}'$$