Multipole expansion

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\vec{r}\,'$$

Let $r >> r' \;\forall\; \vec{r}\,'$ in Ω

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}}$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'\right]^{-1/2} \qquad \frac{r'}{r} << 1$$

$$\approx \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{r'}{r}\cos\theta' + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \cdots\cdots\right]$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{\Omega}(r')^n P_n(\cos\theta')\rho(\vec{r}\,')d\vec{r}\,'$$

2

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int_{\Omega}\rho(\vec{r}\,')d\vec{r}\,' + \frac{1}{r^2}\int_{\Omega} r'\cos\theta'\rho(\vec{r}\,')d\vec{r}\,'\right.$$

$$+\frac{1}{r^3}\int \rho(\vec{r}\,')\, r'^2\left(\frac{3\cos^2\theta'-1}{2}\right)d\vec{r}\,'$$

$$\left. + \cdots\cdots \right]$$

$$q = \int_{\Omega}\rho(\vec{r}\,')d\vec{r}\,'$$

$$r'\cos\theta' = \hat{r}\cdot\vec{r}\,' ; \quad \vec{p} = \int_{\Omega}\vec{r}\,'\rho(\vec{r}\,')d\vec{r}\,'$$

$$r'^2(3\cos^2\theta'-1) = 3r'^2\cos^2\theta' - r'^2$$

$$= 3(r'\cos\theta')(r'\cos\theta') - r'^2$$

$$= 3(r'_i n_i)(r'_j n_j) - r'^2\delta_{ij}n_i n_j$$

$$= A_{ij}n_i n_j \quad [A_{ij} = 3r'_i r'_j - r'^2]$$

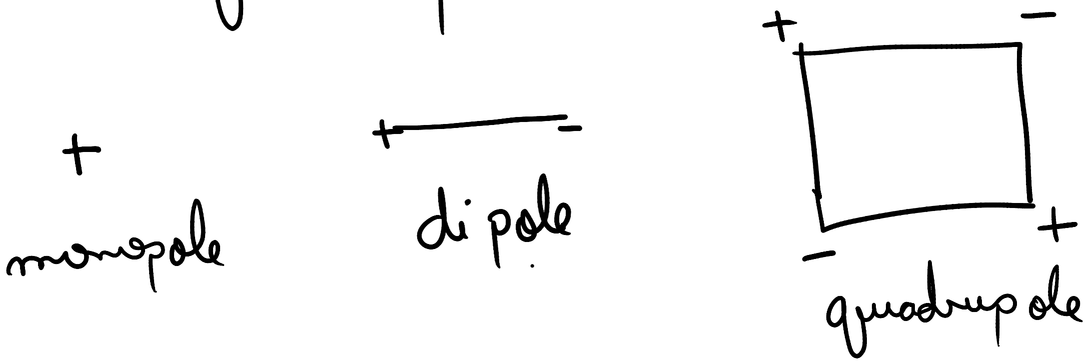
$$[\vec{r}\,' = r'_1\hat{e}_1 + r'_2\hat{e}_2 + r'_3\hat{e}_3] ; \quad \hat{r} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$$

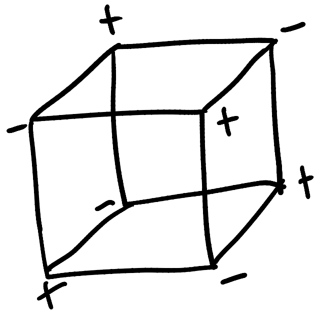
$$Q_{ij} = \int(3r'_i r'_j - r'^2\delta_{ij})d\vec{r}\,'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{2}\,\hat{r}\cdot\overleftrightarrow{Q}\cdot r + \cdots\right]$$

$\overleftrightarrow{Q}$ has components given by Q_{ij}

$\overleftrightarrow{Q}$ - quadrupole moment tensor





Octopole

Energy of interaction with electric potential

$$\mathcal{E} = \int_{\Omega} \rho(\vec{r})\,\varphi(\vec{r})\,d\vec{r}$$

$$\varphi(\vec{r}) = \varphi(0) + \vec{r}\cdot\vec{\nabla}\varphi(0) + \frac{1}{2}\, r_i \frac{\partial^2 \varphi(0)}{\partial r_i\,\partial r_j}\, r_j + \cdots$$

$$\varphi(\vec{r}) = \varphi(0) - \vec{r}\cdot\vec{E}(0) - \frac{1}{2}\, r_i \frac{\partial E_i(0)}{\partial r_j}\,\partial r_j + \cdots$$

$$\varphi(\vec{r}) = \varphi(0) - \vec{r}\cdot\vec{E}(0) - \frac{1}{6}\left(3r_i r_j - r^2\delta_{ij}\right)\frac{\partial E_i(0)}{\partial r_j} + \cdots$$

$\vec{\nabla}\cdot\vec{E} = 0$ (external field)

$$\Rightarrow \frac{1}{6} r^2\, \vec{\nabla}\cdot\vec{E} = 0$$

$$\mathcal{E} = q\,\varphi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6} Q_{ij}\frac{\partial E_i(0)}{\partial r_j} + \cdots$$

Field of a dipole

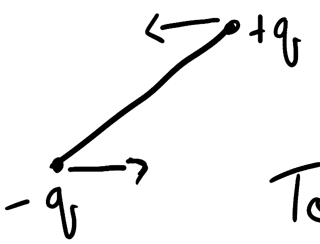
$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} \quad ; \quad p = p\,\hat{z}$$

$$E = -\frac{\partial\varphi}{\partial r} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Energy of interaction between two dipoles

$$\mathcal{E}_{12} = \frac{\vec{p}_1\cdot\vec{p}_2 - 3(\hat{r}\cdot\vec{p}_1)(\hat{r}\cdot\vec{p}_2)}{4\pi\varepsilon_0 r^3} \quad ; \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

Force on a dipole



$\vec{F} = 0$

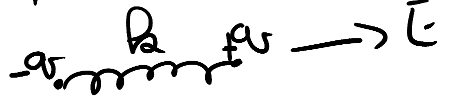
Torque $\vec{\tau} = \vec{p}\times\vec{E}$

Non-uniform electric field

$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

polarizability

$$\vec{p} = \alpha \vec{E}$$



$$q E = k x$$

$$x = \frac{q}{k} E$$

$$\vec{p} = q x = \frac{q^2}{k} \vec{E}$$

induced dipole moment

$\vec{P} = N \vec{p}$ N- # per unit volume

polarizability

$\vec{P} \propto \vec{E}$; $\vec{P} = \varepsilon_0 \chi \vec{E}$ susceptibility

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \frac{\hat{\mathscr{r}} \cdot \vec{P}(\vec{r}\,')\, d\vec{r}\,'}{\mathscr{r}^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\Omega} P(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\vec{r}\,'$$

$$\left[\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'\right]$$

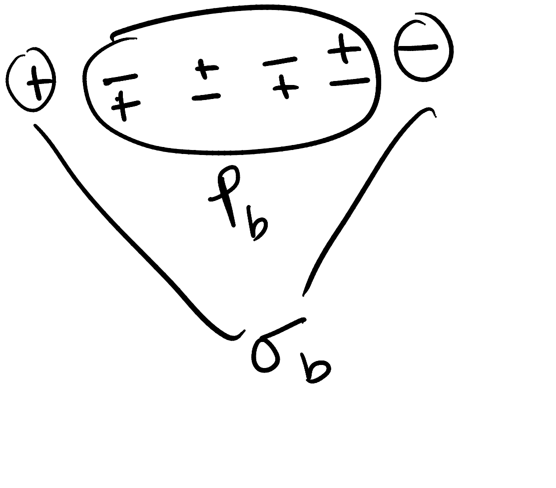
$$= \frac{1}{4\pi\varepsilon_0}\int_{\Omega}\vec{\nabla}'\cdot\left(\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\right)d\vec{r}\,'$$

$$- \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\left(\vec{\nabla}'\cdot\vec{P}'\right)d\vec{r}\,'$$

$$= \frac{1}{4\pi\varepsilon_0}\oint_{\partial\Omega}\frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\cdot d\vec{S} - \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P}'d\vec{r}\,'$$

Bound surface charge

$$\sigma_b = \vec{P}\cdot\hat{n}\,;\quad \vec{\nabla}'\cdot\vec{P} = -\rho_b$$



$$\rho_{tot} = \rho_{free} + \rho_b$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_{free}}{\varepsilon_0} - \frac{\vec{\nabla} \cdot \vec{P}}{\varepsilon_0}$$

$$\vec{\nabla} \cdot (\vec{P} + \vec{\varepsilon}_0 \vec{E}) = \rho_{free}$$

$$\vec{D} = \vec{P} + \varepsilon_0 \vec{E}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

$$\vec{\nabla} \times \vec{D} \neq 0 \quad \because \vec{\nabla} \times \vec{P} \neq 0$$

Thus $\vec{D} \neq \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_{free}(\vec{r}\,') \hat{\mathscr{r}} \, d\vec{r}\,'}{\mathscr{r}^2}$