Laplace's Equation

Polar coordinates

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \varphi^2} = 0$$

$$V = R(r)\,\Phi(\varphi)$$

$$\frac{d^2\Phi}{d\varphi^2} = -\nu^2 \Phi$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{\nu^2}{r^2} R = 0$$

$$R(r) = r^\alpha$$

$$\alpha = \pm \nu$$

$$\Phi(\varphi) = A\cos\nu\varphi + B\sin\nu\varphi$$

ν - integer

$$R(r) = C_\nu r^\nu + D_\nu r^{-\nu} \quad \text{for } \nu \neq 0$$

$$= C_0 + D_0 \ln r \quad \text{for } \nu = 0$$

Cylindrical coordinates - Bessel functions

Complex analysis

$z = x + iy$

$f(z) = u(x,y) + i\,v(x,y)$

$f(z)$ - analytic at a point z_0 if $f'(z_0)$ exists in a neighbourhood around z_0

Cauchy - Reimann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Analytic mappings - Harmonic functions

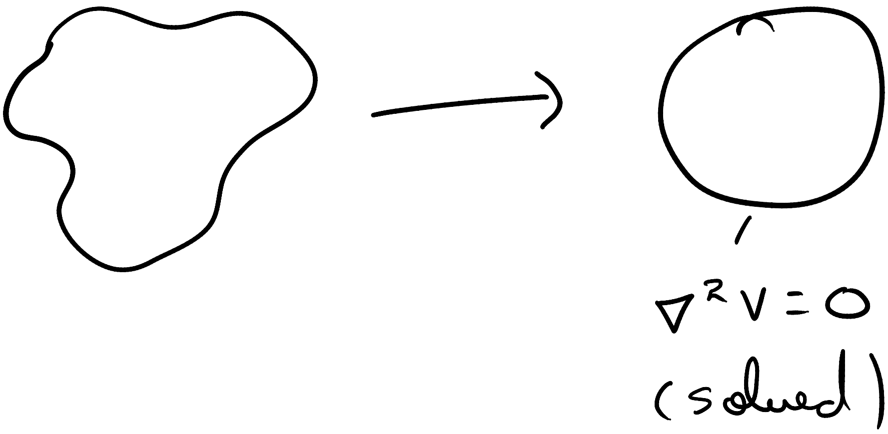
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$(x, y) \to (u, v)$

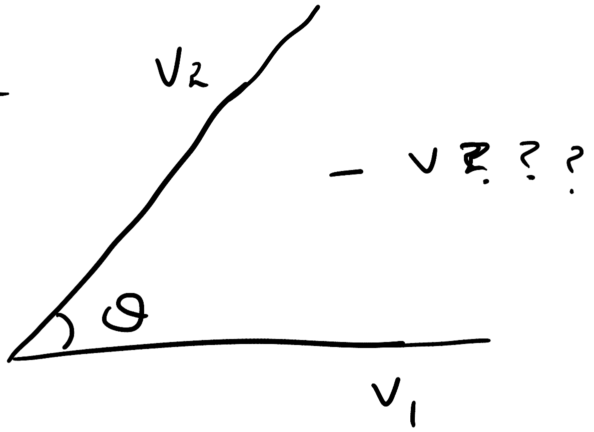
(Conformal transformation)

$$\Rightarrow \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial \omega^2} = 0$$

Idea.



Example



$\nabla^2 V = 0$

$V(y=0) = V_1$; $x \in [0, \infty]$

$V(y = x \tan\theta) = V_2$; $x \in [0, \infty]$

$z = x + iy = Re^{i\phi}$

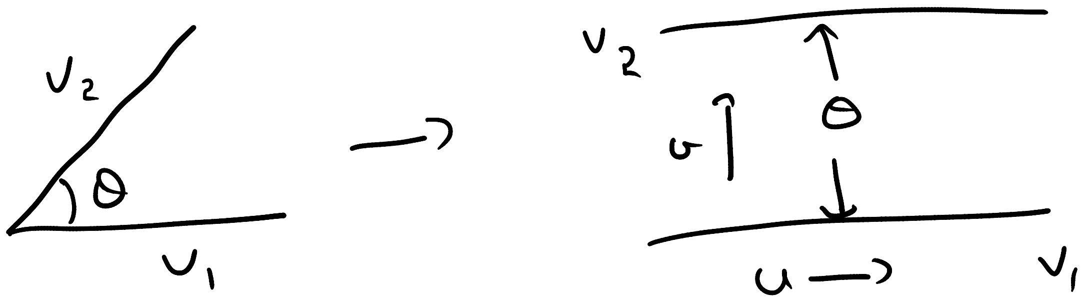
$\omega = \ln(z)$; Show that this transformation is conformal.

$\omega = \ln R + i\phi$; $\phi = 0 \Rightarrow y = 0$

$= u + iv$ $\qquad \omega = -\infty$ to $+\infty$

$\phi = \theta \Rightarrow y = x\tan\theta$

$\omega = -\infty + i\theta$ to $\infty + i\theta$



V is independent of u and linear in v

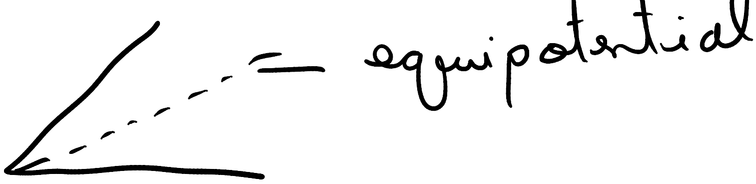
$$V = Av + B$$

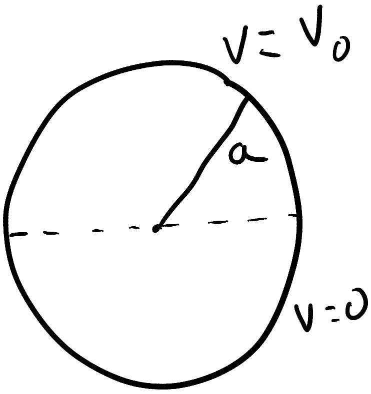
$$B = V_1 ; \quad A = \frac{V_2 - V_1}{\theta}$$

$$V = \frac{V_2 - V_1}{\theta} v + V_1$$

$$v = \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$V(x,y) = \frac{V_2 - V_1}{\theta} \tan^{-1}\left(\frac{y}{x}\right) + V_1$$





$z = Re^{i\varphi}$

Conformal transformation 1

$$\omega_1 = i\left(\frac{a-z}{a+z}\right)$$

$u_1 + i v_1 = \omega_1$

$$\Rightarrow u_1 = \frac{2aR\sin\varphi}{(a^2+R^2+2aR\cos\varphi)}; \quad v_1 = \frac{a^2-R^2}{R^2+a^2+2aR\cos\varphi}$$

x axis $\Rightarrow \sin\varphi = 0 \quad u_1 = 0$

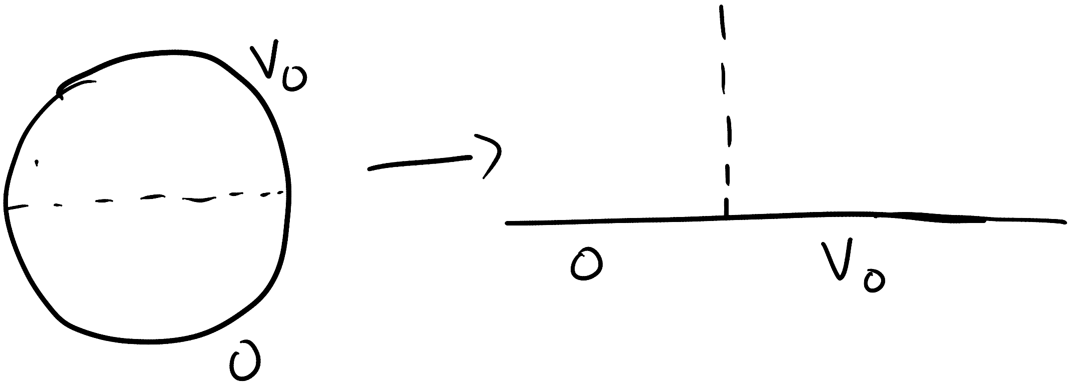
$$v_1 = \frac{a-R}{a+R} \text{ for } x>0$$

$$= \frac{a+R}{a-R} \text{ for } x<0$$

v_1 goes from 1 to 0 for R from 0 to a for $x>0$

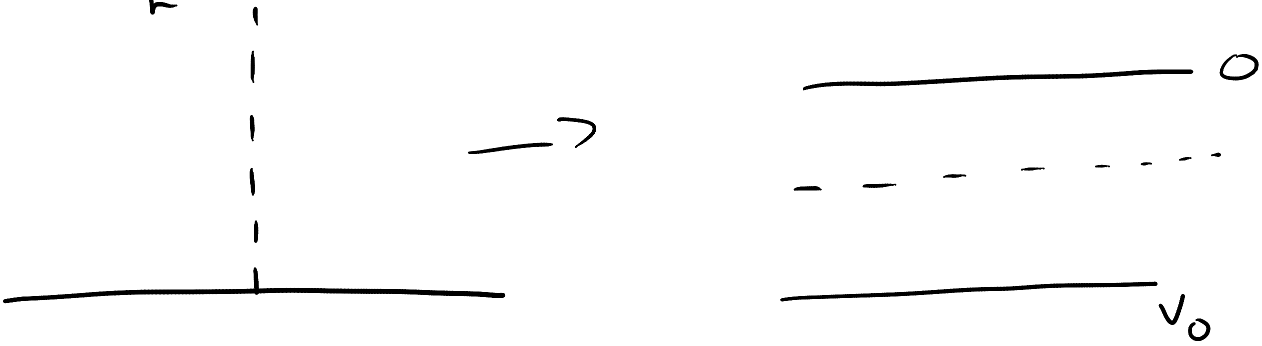
v_1 goes from 1 to ∞ for R from 0 to a for $x<0$

$z = ae^{i\varphi}; \quad \omega_1 = \tan\frac{\varphi}{2}$



Second conformal transformation

$$\omega_2 = \ln(\omega_1) = u_2 + i v_2$$



$$V = -\frac{V_0}{\pi} v_2 + V_0$$

$$v_2 = \arg(\omega_1) = \tan^{-1}\left(\frac{a^2-R^2}{2aR\sin\phi}\right)$$

$$V(R,\phi) = V_0\left[1-\frac{1}{\pi}\tan^{-1}\left(\frac{a^2-R^2}{2aR\sin\phi}\right)\right]$$

Equipotential $\dfrac{a^2-R^2}{2aR\sin\phi} =$ cons