Scalar product

$$B \cdot A \equiv B_\alpha A^\alpha$$

$$B' \cdot A' = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} B_\beta A^\gamma$$

$$= \delta^\beta_\gamma B_\beta A^\gamma = B_\beta A^\beta = B \cdot A$$

Invariant

$$ds^2 = (dx^0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2$$

$$= g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$g_{00} = 1 \;;\; g_{11} = g_{22} = g_{33} = -1$$

$g_{\alpha\beta}$ - metric tensor

$$g'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta}$$

$g_{\alpha\beta} = g^{\alpha\beta}$ (for flat space time)

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta^\beta_\alpha$$

$x_\alpha = g_{\alpha\beta} x^\beta$ $\quad A^\alpha = (A^0\ \vec{A})$

$x^\alpha = g^{\alpha\beta} x_\beta$ $\quad A_\alpha = (A^0, -\vec{A})$

$$F^{\cdots\alpha\cdot}_{\cdot\,\cdot\,\cdots} = g^{\alpha\beta} F^{\cdots\cdots\beta\cdot}_{\cdots}$$

|||ly $G^{\cdots\cdots}_{\cdot\cdot\alpha} = g_{\alpha\beta} G^{\cdots\cdot\beta}_{\cdots\cdot}$

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}}$$

$\frac{\partial}{\partial x^{\alpha}}$ transforms as a covariant vector

$\frac{\partial}{\partial x_{\alpha}}$ transforms as a contravariant vector

$$\partial^{\alpha} \equiv \frac{\partial}{\partial x_{\alpha}} = \left(\frac{\partial}{\partial x^{0}}, -\vec{\nabla}\right)$$

$$\partial_{\alpha} \equiv \frac{\partial}{\partial x^{\alpha}} = \left(\frac{\partial}{\partial x^{0}}, \vec{\nabla}\right)$$

$$\partial^{\alpha} A_{\alpha} = \partial_{\alpha} A^{\alpha} = \frac{\partial A^{0}}{\partial x^{0}} + \vec{\nabla} \cdot \vec{A}$$

four divergence

$$\square^{2} = \partial_{\alpha} \partial^{\alpha} = \frac{\partial^{2}}{\partial x^{0\,2}} - \nabla^{2}$$

Kinematics

In a particle's rest frame $d\vec{x} = 0$

$$c^{2} (d\tau)^{2} = c^{2} dt^{2} - (dx^{1})^{2} - (dx^{2})^{2} - (dx^{3})^{2}$$

$d\tau$ - Lorentz scalar - proper time

Four velocity

$$v^\alpha = \frac{dx^a}{d\tau}$$ – Contravariant vector

Four momentum

$$p^\alpha = m_0 \frac{dx^\alpha}{d\tau}$$

Assume relative $\vec{v} = v\hat{x}$

$$\frac{dt}{d\tau} = \gamma \;;\; \frac{dx}{d\tau} = \gamma v \;;\; \frac{dy}{d\tau} = \frac{dz}{d\tau} = 0$$

$$p^\alpha = (\gamma m_0 c, \gamma m_0 v, 0, 0)$$

$$p^x = \gamma m_0 v = mv, \quad m = \gamma m_0$$

$$p^\mu p_\mu = -(p^0)^2 + \vec{p}\cdot\vec{p} = -m_0^2 c^2$$

Force

$$\frac{d\vec{p}}{dt} = \vec{F} \;;\; \vec{p} = m_0 \frac{d\vec{x}}{dt}$$

$$\int \vec{F}\cdot d\vec{r} = T \text{ (Kinetic energy)}$$

$$dT = \vec{v}\cdot d\vec{p}$$

$$T = \int_{\vec{v}=0}^{\vec{v}} d(\vec{p}\cdot\vec{v}) - \int_{\vec{v}=0}^{\vec{v}} \vec{p}\cdot d\vec{v}$$

$\vec{p} = \gamma m_0 \vec{u}$

$$T = \gamma m_0 u^2 - \frac{m_0}{2}\int_0^{u^2} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\, du^2$$

$$= \gamma m_0 u^2 + m_0 c^2 \sqrt{1-\frac{u^2}{c^2}} - m_0 c^2$$

$$= \frac{m_0 u^2}{\sqrt{1-\frac{u^2}{c^2}}} + m_0 c^2 \sqrt{1-\frac{u^2}{c^2}} - m_0 c^2$$

$$= \gamma m_0 c^2 - m_0 c^2$$

$\gamma m_0 c^2 = T + m_0 c^2$ → Rest energy

Total Energy

$m = \gamma m_0$

$\underline{E = mc^2} = p^0 c$

$-(p^0)^2 + |\vec{p}|^2 = -m_0^2 c^2$

$\Rightarrow E^2 = \sqrt{|\vec{p}|^2 c^2 + m_0^2 c^4}$

Four force

$$F^\mu = \frac{dp^\mu}{dt}$$