Special theory of relativity ①

Newton's laws invariant under Galilean transformations

If Newton's laws hold in some frame of reference, they will hold in another frame of reference defined by the transformations

$$\vec{r}' = \vec{r} - \vec{v}t$$
$$t' = t$$

Primed frame has uniform velocity $\vec{v}$ as seen from unprimed frame

Wave equation not invariant under Galilean transformations

Let $\vec{v} = v\hat{z}$

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0.$$

$$\rightarrow \left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{2}{c^2}\vec{v}\cdot\vec{\nabla}'\frac{\partial}{\partial t'} - \frac{1}{c^2}(\vec{v}\cdot\vec{\nabla}')^2\right]\psi' = 0$$

No "simple" (kinematic) transformation on ψ to restore invariance.

Postulates of relativity

1) Laws of nature and results of experiments in a frame of reference invariant under uniform translation of the frame

2) Speed of light is independent of the motion of its source

2') Universal limiting speed c.

Linear transformations

Assume that two frames have relative velocity $\vec{v} = v\hat{x}$

$$x' = A(v)x + B(v)t$$

$$t' = C(v)x + D(v)t$$

Limiting universal speed $c \Rightarrow$

$$\left[\frac{dx'}{dt'} = \pm c \Leftrightarrow \frac{dx}{dt} = \pm c \right]$$

$$\Rightarrow \alpha(v)\left[c^2 dt^2 - dx^2\right] = c^2 (dt')^2 - (dx')^2$$

primed $\longleftrightarrow$ unprimed

$$\Rightarrow \alpha(v) = \frac{1}{\alpha(-v)}$$

Further isotropy $\Rightarrow \alpha(v) = \alpha(-v)$

$\Rightarrow \alpha(v)^2 = 1$ or $\alpha(v) = \pm 1$

$$c^2 (dt')^2 - (dx')^2 = \pm [c^2 dt^2 - dx^2]$$

$$dx' = 0 \Rightarrow \frac{dx}{dt} = v$$

$$(dt')^2 = \pm \frac{1}{c^2} \left[c^2 - v^2 \right] (dt)^2$$

$v < c \Rightarrow +$ sign has to be chosen so that t and t' remain real

$$(dt')^2 = \left(1 - \frac{v^2}{c^2}\right)(dt)^2$$

$$dt = \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$ (+ sign so that $dt = dt'$ when $v \to 0$)

$$dx = \frac{v\, dt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow t = \gamma [t' + f(x')] \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x = \gamma [v t' + g(x')]$$

$$\frac{dx}{dt} = c = \frac{v + g' \frac{dx'}{dt'}}{1 + f' \frac{dx'}{dt'}} = \frac{v + g' c}{1 + f' c}$$

$$\because \frac{dx'}{dt'} = c$$

$\Rightarrow v + g'c = c + f'c^2$

or $g' = 1 - \frac{v}{c} + f'c$

$dt' = 0$

$\Rightarrow dx = \gamma g' dx'$

$dt = \gamma f' dx'$

$\gamma^2 (dx')^2 [c^2 (f')^2 - (g')^2] = -(dx')^2$

or $g'^2 - c^2 (f')^2 = \frac{1}{\gamma^2}$

$g' + cf' = \frac{1}{\gamma^2} \frac{1}{1 - v/c} = 1 + v/c$

$g' - cf' = 1 - v/c$

$g' = 1 \;;\; f' = v/c^2$

If $(x', t') = (0,0) \Rightarrow (t, x) = (0,0)$

$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

$x = \gamma(vt' + x')$

$\Rightarrow x' = \gamma(x - vt)$

$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

Lorentz transformations

$\beta = v/c$

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$x_0 = ct$$
$$\vec{x} = (x_1, x_2, x_3)$$

$$x_0' = \gamma(x_0 - \vec{\beta}\cdot\vec{x})$$
$$\vec{x}' = \vec{x} + \frac{\gamma - 1}{\beta^2}(\vec{\beta}\cdot\vec{x})\beta - \gamma\beta x_0$$

$$\vec{\beta} = \frac{\vec{v}}{c}$$

$\beta = \tanh \rho$ $\quad [\beta \in [0,1]]$

$[\gamma \in [0, \infty]]$

$$\gamma = \cosh \rho$$
$$x_0' = x_0 \cosh\rho - x_1 \sinh\rho$$
$$x_1' = -x_0 \sinh\rho + x_1 \cosh\rho$$

Like rotation matrix but cosh & sinh in place of cos and sin

$x_0 \to i x_0$ and $v \to iv$ restores cos & sin

Four vector

$$A_0' = \gamma(A_0 - \vec{\beta}\cdot\vec{A})$$
$$A_{\parallel}' = \gamma(A_{\parallel} - \beta A_0)$$
$$A_\perp' = A_\perp$$
$$(A_0')^2 - |\vec{A}'|^2 = A_0^2 - |\vec{A}|^2$$

Scalar product

$$A_0' B_0' - \vec{A}' \cdot \vec{B}' = A_0 B_0 - \vec{A} \cdot \vec{B}$$

preserved by Lorentz transformations

$$x'^\alpha = x'^\alpha(\{x^\beta\}) \qquad (\alpha, \beta = 0, 1, 2, 3)$$

Lorentz scalar unchanged by Lorentz transformations

Contravariant vector

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta$$

Covariant vector

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta$$

Contravariant tensor of rank 2

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$$

Covariant tensor of rank 2

$$G'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} G_{\gamma\delta}$$

Mixed tensor of rank 2

$$H'^\alpha{}_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} H^\gamma{}_\delta$$