Current carrying wire

+ charges $= v\hat{z}$, − charges $= -v\hat{z}$

$\lambda_+ = \lambda_- = \lambda$ $\Big\}$ Lab frame

$\lambda_{tot} = \lambda_+ - \lambda_- = 0$

$\vec{I} = 2\lambda v\hat{z}$

$\vec{E} = 0$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

$$\phi = 0, \quad A_z = -\frac{\mu_0 I}{2\pi}\log r$$

Charge q_1 moving with velocity $u\hat{z}$

In its rest frame

$$v_\pm = \frac{v \mp u}{1 \mp \frac{u}{c^2}} \; ; \; \gamma_\pm = \frac{1}{\sqrt{1-\left(\frac{v_\pm}{c}\right)^2}}$$

Length contraction of intercharge distance

$$\lambda'_{\pm} = \gamma_{\pm} \lambda_0$$ λ_0 – charge density in rest frame of charges

$$\lambda'_{tot} = \lambda'_{+} - \lambda'_{-} = \lambda_0(\gamma_+ - \gamma_-)$$

$$\lambda = \gamma \lambda_0 \,, \quad \lambda'_{tot} = \lambda\left(\frac{\gamma_+ - \gamma_-}{\gamma}\right)$$

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\lambda'_{tot} = -\frac{2\lambda u v}{c^2\sqrt{1-\frac{u^2}{c^2}}}$$

Negative charge on wire

Electric field $\vec{E} = \dfrac{\lambda'_{tot}}{2\pi\varepsilon_0 r}\hat{r}$

$$= -\frac{\lambda v}{\pi \varepsilon_0 c^2 r}\frac{u}{\sqrt{1-u^2/c^2}}\hat{r}$$

If charge q on particle

$$\vec{F}' = -\frac{\mu_0 I}{2\pi r}\frac{q u}{\sqrt{1-\frac{u^2}{c^2}}}\hat{r}$$

In lab frame

$$\vec{F} = \sqrt{1-\frac{u^2}{c^2}}\,\vec{F}'$$

$$= -\frac{\mu_0 I}{2\pi r} q u = q(\vec{u}\times\vec{B})$$

$$\lambda'_{tot} = \frac{-\frac{u}{c^2} I}{\sqrt{1-\frac{u^2}{c^2}}} = -\frac{2\lambda u v}{\sqrt{1-\frac{u^2}{c^2}}}$$ from Lorentz transformations

agrees with previous result

$$\phi'(r) = \frac{-u A_z}{\sqrt{1-\frac{u^2}{c^2}}} = -\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\mu_0 \lambda v}{\pi}\log r$$

$$\vec{E} = -\vec{\nabla}'\phi' = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\frac{u}{r\sqrt{1-\frac{u^2}{c^2}}}\hat{r}' \quad (\because r = r')$$

Langrangian formulation of relativistic mechanics

Non-relativistic

$L = T + V$

$S = \int L\, dt$

Relativistic

$S = \int L\, dt = \int \gamma L\, d\tau$

S invariant $\Rightarrow \gamma L$ invariant

Free particle

Lorentz invariant $u^{\alpha} u_{\alpha} = -c^2$

$\gamma L \propto -c^2 \Rightarrow \gamma L = -m_0 c^2$

$L = -m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\Rightarrow \frac{d}{dt}(\gamma m u) = 0$$

In the presence of EM fields

Non-relativistic

$L = T - V \;\; ; \;\; V = q\phi$

Relativistic should reduce to $q\phi$

$L = L_{free} + L_{int}$

γL_{int} - Lorentz invariant

$L_{int} \to -q\phi$ when $\gamma \to 1$

$$-q\phi = -qc\frac{\phi}{c} = -qcA^0 = -q\,u_0 A^0$$

$$L_{int} = -\frac{q}{\gamma} u_\alpha A^\alpha = -q\phi + q\vec{v}\cdot\vec{A}$$

$$L = -m_0 c^2\sqrt{1 - \frac{u^2}{c^2}} - q\phi + q\vec{v}\cdot\vec{A}$$

$$P_i = \frac{\partial L}{\partial v_i} = \gamma m_0 v_i + q A_i = p_i + q A_i$$

$$p_i = P_i - q A_i \;\; ; \;\; \vec{p} = \vec{P} - e\vec{A}$$

mechanical momentum

Hamiltonian

$$H = \vec{P} \cdot \vec{v} - L$$

$$\vec{v} = \frac{c(\vec{P} - e\vec{A})}{\sqrt{(\vec{P} - e\vec{A})^2 + m_0^2 c^2}}$$

$$H = \sqrt{(\vec{P} - e\vec{A})^2 c^2 + m^2 c^4} + q\phi$$

$$H = \omega$$

$$(\omega - q\phi)^2 - (\vec{P} - e\vec{A})^2 c^2 = m^2 c^4$$

$$p^a p_\alpha = (mc)^2$$

$$p^a = \left(\frac{E}{c}, \vec{P}\right) = \left(\frac{1}{c}(\omega - q\phi), \vec{P} - e\vec{A}\right)$$

$$p^\alpha = \left(\frac{\omega}{c}, \vec{P}\right)$$