Relativistic E & M

Charge conservation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

$$\partial_\mu J^\mu = 0$$

$$\partial'_\nu J'^\nu = \partial_\mu J^\mu$$

$$\partial'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \partial_\mu$$

$$\frac{\partial x^\mu}{\partial x'^\nu} \partial_\mu J'^\nu = \partial_\mu J^\mu$$

$$\partial_\mu \left[\frac{\partial x^\mu}{\partial x'^\nu} J'^\nu \right] = \partial_\mu J^\mu$$

$$\Rightarrow J^\mu = \frac{\partial x^\mu}{\partial x'^\nu} J'^\nu$$

$$\frac{\partial x'^\sigma}{\partial x^\mu} J^\mu = \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\nu} J'^\nu$$

$$J'^\sigma = \frac{\partial x'^\sigma}{\partial x^\mu} J^\mu$$

J is a 4-vector

Lorentz gauge

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t}$$

$$\Rightarrow \partial_\mu A^\mu = 0$$

$$A^\mu = \left(\frac{\varphi}{c}, A_x, A_y, A_z\right)$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial_2 A^3 - \partial_3 A^2 = -\left[\partial^2 A^3 - \partial^3 A^2\right]$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \partial_3 A^1 - \partial_1 A^3 = -\left[\partial^1 A^3 - \partial^3 A^1\right]$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \partial_1 A^2 - \partial_2 A^1 = -\left[\partial^1 A^2 - \partial^2 A^1\right]$$

$$E_x = -\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t} = -c\left(\partial_1 A^0 + \partial_0 A^1\right) = c\left(\partial^1 A^0 - \partial^0 A^1\right)$$

$$E_y = -\frac{\partial \varphi}{\partial y} - \frac{\partial A_y}{\partial t} = -c\left(\partial_2 A^0 + \partial_0 A^2\right) = c\left(\partial^2 A^0 - \partial^0 A^2\right)$$

$$E_z = -\frac{\partial \varphi}{\partial z} - \frac{\partial A_z}{\partial t} = -c\left(\partial_3 A^0 + \partial_0 A^3\right) = c\left(\partial^3 A^0 - \partial^0 A^3\right)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$F^{\mu\nu}$ - second rank antisymmetric contravariant tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = g_{\alpha\mu} g_{\beta\nu} F^{\alpha\beta}$$

$F_{\mu\nu}$ can be obtained from $F^{\mu\nu}$ by $\vec{E} \to -\vec{E}$ and $\vec{B} \to \vec{B}$

Dual field tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$\varepsilon^{\alpha\beta\gamma\delta} = 0$ if any two indices are equal

changes sign under an interchange of any two indices

$\varepsilon^{0123} = 1$

$\tilde{F}^{\mu\nu}$ can be obtained from $F^{\mu\nu}$ with $\vec{E} \to \vec{B}c$ and $\vec{B} \to -\vec{E}/c$

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \;;\; \vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$E_i = cF^{i0} \quad (i=1,2,3)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$c\frac{\partial F^{i0}}{\partial x_i} = \frac{\rho}{\varepsilon_0} = \frac{J^0}{c\varepsilon^0}$$

or
$$\frac{\partial F^{i0}}{\partial x_i} = \mu_0 J^0$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

gives both $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ & $\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$

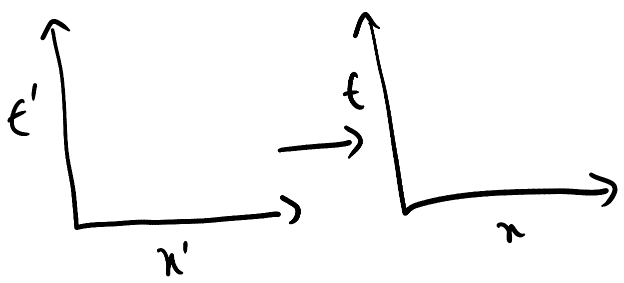
$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$\vec{\nabla}\cdot\vec{B} = 0$ & $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\boxed{\begin{aligned}\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0\end{aligned}}$$

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}F^{\rho\sigma}$$



$E_1' = E_1$; $B_1' = B_1$

$E_2' = \gamma(E_2 - \beta B_3 c)$; $B_2' = \gamma(B_2 + \beta \frac{E_3}{c})$

$E_3' = \gamma(E_3 + \beta B_2 c)$; $B_3' = \gamma(B_3 - \beta \frac{E_2}{c})$

In general

$$\vec{E} = \gamma(\vec{E} + \vec{\beta} \times \vec{B}c) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B} = \gamma(\vec{B} - \vec{\beta} \times \frac{\vec{E}}{c}) - \frac{\gamma^2}{\gamma+1} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

$\vec{E} \cdot \vec{B}$ is a Lorentz scalar as is

$\frac{E^2}{c^2} - B^2$.

Force on a charged particle
(Connection with mechanics)

Charge's instantaneous rest frame

$\frac{d\vec{p}}{dt} = q\vec{E}$ (has to agree with Newtonian result)

Further, in the same frame

$\frac{d\varepsilon}{dt} = \vec{F} \cdot \vec{v} = 0$ ($\because \vec{v} = 0$)

Relativistic generalization

$\frac{E_\alpha}{c} = F^{\alpha 0} \qquad (\alpha = 1, 2 \& 3)$

$$\left.\frac{dp_\alpha}{dt}\right|_{rest} = \left.\frac{dp_\alpha}{d\tau}\right|_{rest} = qc\,F^{\alpha 0}$$

$c = v_0$

$$\left.\frac{dp_\alpha}{d\tau}\right|_{rest} = q\, v_0\, F^{\alpha 0}\Big|_{rest}$$

$$\left.\frac{d\varepsilon}{dt}\right|_{rest} = 0 = q\, v_\alpha F^{0\alpha}\Big|_{rest} \quad (\because v_\alpha = 0)$$

These two equations can be combined to give

$$\frac{dp^\mu}{d\tau} = q\, U_\nu F^{\mu\nu} \text{ – Minkowski force}$$

$$K^\mu = m\frac{dU^\mu}{d\tau} = q\, U_\nu F^{\mu\nu}$$