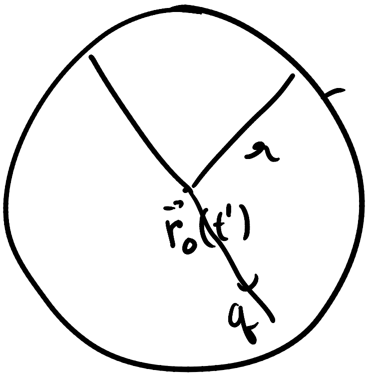
Radiation from a point charge - 2

Poynting vector

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0 c}\left[\vec{E}\times(\hat{r}\times\vec{E})\right]$$

$$= \frac{1}{\mu_0 c}\left[\hat{r}E^2 - E(\hat{r}\cdot\vec{E})\right]$$



Integrate S over sphere with $r \to \infty$

Area of sphere $\propto r^2$, so $\vec{S} \propto \frac{1}{r^2}$ will yield a finite answer but not $\vec{S} \propto \frac{1}{r^3}$ or $\frac{1}{r^4}$ or....

Only acceleration fields have $\vec{E}, \vec{B} \propto \frac{1}{r}$

so $\vec{S} \propto \frac{1}{r^2}$

$$\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{r}{(\vec{r}\cdot\vec{u})^3}\left[\vec{r}\times(\vec{u}\times\vec{a})\right]$$

$$\hat{r}\cdot\vec{E}_{rad} = 0$$

$$\vec{S} = \frac{1}{\mu_0 c}E_{rad}^2\,\hat{r}$$

Simple case

$\vec{v}(t_r) = 0 \Rightarrow \vec{u} = c\hat{\imath}$

So $\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{\imath}\left[\hat{\imath}\times(\hat{\imath}\times\vec{a})\right]$

$= \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{\imath}\left[\hat{\imath}(\hat{\imath}\cdot\vec{a}) - \vec{a}\right]$

$\vec{S}_{rad} = \frac{1}{\mu_0 c}\left(\frac{q}{4\pi\varepsilon_0 c^2}\right)^2 \frac{1}{\imath^2}\left[a^2 - (\hat{\imath}\cdot a)^2\right]\hat{\imath}$

Let $\vec{a}$ be along the z direction

$\vec{S}_{rad} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c^3}\frac{a^2\sin^2\theta}{\imath^2}\hat{\imath}$

$\vec{P} = \oint \vec{S}_{rad}\cdot d\vec{a} = \frac{1}{\varepsilon_0 c}\left(\frac{q}{4\pi c}\right)^2 a^2 \int \frac{\sin^2\theta}{\imath^2}\imath^2 \sin\theta \, d\theta \, d\phi$

$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}$ - Larmor formula

If $\vec{v}(t') \neq 0$

Rate at which energy passes through sphere

$\neq$ rate at which it left particle (Doppler effect)

$\frac{dW}{dt'} = \frac{dW}{dt} \Big/ \frac{\partial t'}{\partial t} = \frac{\vec{\imath}\cdot\vec{u}}{\imath c}\frac{dW}{dt}$

particle   sphere

$$\frac{\vec{r}\cdot\vec{u}}{rc} = \left(1 - \frac{\hat{r}\cdot\vec{v}}{c}\right)$$

$$\frac{dP_{rad}}{d\Omega} = \frac{\vec{r}\cdot\vec{u}}{rc}\frac{1}{\mu_0 c}E_{rad}^2 r^2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{4\pi}\right)\frac{[\hat{r}\times(\vec{u}\times\vec{a})]^2}{(\hat{r}\cdot u)^5}$$

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\left[a^2 - \left(\frac{\vec{v}}{c}\times\vec{a}\right)^2\right]\gamma^6$$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$ Liénard-Larmor formula

Radiation reaction ($v \ll c$)

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}$$

Radiation reaction force $\vec{F}_{rad}$

$$\vec{F}_{rad}\cdot\vec{v} = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}$$

Energy lost in interval between t_1 and t_2

$$\int_{t_1}^{t_2}\vec{F}_{rad}\cdot\vec{v}\,dt = -\frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\int_{t_1}^{t_2}a^2\,dt$$

$$\int_{t_1}^{t_2}a^2\,dt = \int_{t_1}^{t_2}\frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}\,dt = \vec{v}\cdot\frac{d\vec{v}}{dt}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d^2\vec{v}}{dt^2}\cdot\vec{v}\,dt$$

Assume periodic motion t_1 and t_2 are endpoints of a period so $\vec{v}\cdot\frac{d\vec{v}}{dt}\Big|_{t_1}^{t_2} = 0$

$$\Rightarrow \int_{t_1}^{t_2} \left[\vec{F}_{rad} - \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}\right]\cdot\vec{v}\,dt = 0$$

A plausible definition of $\vec{F}_{rad}$ is

$$\vec{F}_{rad} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{q^2}{c^3}\dot{\vec{a}}$$

Abraham-Lorentz formula

Heuristic - Only for periodic motion

Example: simple pendulum

$$m\ddot{x} = -kx + m\tau\dddot{x} + F_{driv}$$

$$F_{driv} = F_0\cos(\omega t + \varphi)$$

$$\dddot{x} = -\omega^2\dot{x}$$

Damping $\gamma = \omega^2\tau$