Maxwell's equations in terms of potentials

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$-\nabla^2\varphi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

Lorentz gauge

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$$

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

D'Alembertian operator

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2}$$

Laplacian in 4D with (x, y, z, it)

②

Green's functions for $\Box^2$

$$\Box^2 G(\vec{r}, t; \vec{r}', t') = -4\pi \delta(\vec{r} - \vec{r}')\delta(t-t')$$

Two types of Green's functions

Retarded

$$G_R(\vec{r}, t; \vec{r}', t') = \frac{1}{\mathscr{r}} \delta\left(t - t' - \frac{\mathscr{r}}{c}\right)$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

Advanced

$$G_A(\vec{r}, t; \vec{r}', t') = \frac{1}{\mathscr{r}} \delta\left(t - t' + \frac{\mathscr{r}}{c}\right)$$

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d\vec{r}' \int dt' \, G(\vec{r}, t; \vec{r}', t') \rho(\vec{r}', t')$$

Retarded

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d\vec{r}' \, \frac{\rho(\vec{r}', t - \frac{\mathscr{r}}{c})}{\mathscr{r}} \quad \left[t - \frac{\mathscr{r}}{c} < t\right]$$

Consistent with causality

Advanced

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d\vec{r}' \, \frac{\rho(\vec{r}', t + \frac{\mathscr{r}}{c})}{\mathscr{r}} \quad \left[t + \frac{\mathscr{r}}{c} > t\right]$$

Inconsistent with causality

Only consider retarded Green's functions

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d\vec{r}' \, \frac{\rho(\vec{r}', t + \frac{\mathscr{r}}{c})}{\mathscr{r}}$$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int d\vec{r}' \, \frac{\vec{J}(\vec{r}', t + \frac{\mathscr{r}}{c})}{\mathscr{r}}$$

Liénard - Wiechert potentials

$\varphi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ due to a moving point charge

Let the point charge have a trajectory $\vec{r}(t)$ and charge q

$$\rho(\vec{r},t) = q\,\delta[\vec{r}-\vec{r}_0(t)]$$

$$\vec{J}(\vec{r},t) = q\,\vec{v}(t)\,\delta[\vec{r}-\vec{r}_0(t)]$$

where $\vec{v}(t) = \dfrac{d\vec{r}_0}{dt}$

$$\varphi(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\int d\vec{r}'\,\frac{\delta[\vec{r}'-\vec{r}_0(t-\frac{r}{c})]}{r}$$

$$= \frac{q}{4\pi\varepsilon_0}\int dt'\int d\vec{r}'\,\frac{1}{r}\,\delta[\vec{r}'-\vec{r}_0(t')]\,\delta\left(t'-t+\frac{r}{c}\right)$$

$$\vec{r} = \vec{r}-\vec{r}'$$

$$= \frac{q}{4\pi\varepsilon_0}\int dt'\,\frac{1}{|\vec{r}-\vec{r}_0(t')|}\,\delta\left(t'-t+\frac{|\vec{r}-\vec{r}_0(t')|}{c}\right)$$

$$\tau = t'-t+\frac{|\vec{r}-\vec{r}_0(t')|}{c}$$

$$d\tau = dt' + \frac{1}{c}\,\frac{d|\vec{r}-\vec{r}_0(t')|}{dt'}$$

$$= dt'\left[1-\frac{\hat{n}(t')\cdot\vec{v}(t')}{c}\right]$$

where $\hat{n}(t') = \dfrac{\vec{r} - \vec{r}_0(t')}{|\vec{r} - \vec{r}_0(t')|}$

$$\varphi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\int \frac{d\tau}{1 - \frac{\hat{n}\cdot\vec{v}}{c}} \frac{1}{|\vec{r} - \vec{r}_0(t'(\tau))|}\,\delta(\tau)$$

$$\tau = 0 \Rightarrow t' = t - \frac{|\vec{r} - \vec{r}_0(t')|}{c}$$

or $|\vec{r} - \vec{r}_0(t')| = c(t - t')$

$$\vec{r}_0 \xleftrightarrow{\;c(t-t')\;} \vec{r}$$

$$\varphi(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{c}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}_0\left(t - \frac{\mathscr{r}}{c}\right)$$

$$\vec{v} = \vec{v}\left(t - \frac{\mathscr{r}}{c}\right)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} qc \frac{\vec{v}}{\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v}} = \frac{\vec{v}}{c^2}\varphi(\vec{r},t)$$

What about the fields $\vec{E}$ and $\vec{B}$?

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$$

t' is a function of $\vec{r}$ and t and thus so are $\vec{r}_0$ and $\vec{\mathscr{r}}$

$$\vec{\nabla}\varphi = -\frac{qc}{4\pi\varepsilon_0}\frac{1}{(\mathcal{r}c-\vec{\mathcal{r}}\cdot\vec{v})^2}\vec{\nabla}(\mathcal{r}c-\vec{\mathcal{r}}\cdot\vec{v})$$

$$\mathcal{r} = c(t-t')$$

$$\vec{\nabla}\mathcal{r}c = -c^2\vec{\nabla}t'$$

$$\vec{\nabla}(\vec{\mathcal{r}}\cdot\vec{v}) = (\vec{\mathcal{r}}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{\mathcal{r}} + \vec{\mathcal{r}}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{\mathcal{r}})$$

$$(\vec{\mathcal{r}}\cdot\vec{\nabla})\vec{v}(t') = \mathcal{r}_i\frac{\partial\vec{v}(t')}{\partial x_i} = \frac{d\vec{v}}{dt'}\mathcal{r}_i\frac{\partial t'}{\partial x_i} = \vec{a}(\vec{\mathcal{r}}\cdot\nabla t')$$

$$(\vec{v}\cdot\vec{\nabla})\vec{\mathcal{r}} = (\vec{v}\cdot\vec{\nabla})\vec{r} - (\vec{v}\cdot\vec{\nabla})\vec{r}_0(t')$$

$$(\vec{v}\cdot\vec{\nabla})\vec{r} = v_i\frac{\partial\vec{r}}{\partial x_i} = \vec{v}$$

$$(\vec{v}\cdot\vec{\nabla})\vec{r}_0(t') = \vec{v}(\vec{v}\cdot\vec{\nabla}t')$$

$$(\vec{\nabla}\times\vec{v})_i = \varepsilon_{ijk}\frac{\partial v_k(t')}{\partial x_j} = \varepsilon_{ijk}\frac{\partial t'}{\partial x_j}\frac{dv_k}{dt'}$$

$$\vec{\nabla}\times\vec{v} = \vec{\nabla}t'\times\vec{a}$$

$$\vec{\nabla}\times\vec{\mathcal{r}} = \vec{\nabla}\times\vec{r} - \vec{\nabla}\times\vec{r}_0(t')$$

$$= \vec{v}\times\vec{\nabla}t'$$

$$\vec{\nabla}(\vec{\mathcal{r}}\cdot\vec{v}) = \vec{a}(\vec{\mathcal{r}}\cdot\vec{\nabla}t') + \vec{v} - \vec{v}(\vec{v}\cdot\vec{\nabla}t') + \vec{\mathcal{r}}\times(\vec{\nabla}t'\times\vec{a}) + \vec{v}\times(\vec{v}\times\nabla t')$$

$$= \vec{v} + (\vec{\mathcal{r}}\cdot\vec{a})\vec{\nabla}t' - v^2\nabla t'$$

$$\nabla\varphi = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(\mathcal{r}c-\vec{\mathcal{r}}\cdot\vec{v})^2}\left[\vec{v} + (c^2 - v^2 + \vec{\mathcal{r}}\cdot\vec{a})\nabla t'\right]$$

$$-c\vec{\nabla}t' = \vec{\nabla}r = \vec{\nabla}\sqrt{\vec{r}\cdot\vec{r}} = \frac{1}{2r}\vec{\nabla}(\vec{r}\cdot\vec{r})$$

$$= \frac{1}{r}\left[(\vec{r}\cdot\vec{\nabla})\vec{r} + \vec{r}\times(\vec{\nabla}\times\vec{r})\right]$$

$$(\vec{r}\cdot\vec{\nabla})\vec{r} = \vec{r} - \vec{v}(\vec{r}\cdot\vec{\nabla}t')$$

$$\vec{r}\times(\vec{\nabla}\times\vec{r}) = \vec{r}\times(\vec{v}\times\vec{\nabla}t')$$

$$\vec{\nabla}r = \frac{1}{r}\left[\vec{r} - (\vec{r}\cdot\vec{v})\nabla t'\right] = -c\vec{\nabla}t'$$

$$\Rightarrow \vec{\nabla}t' = \frac{-r}{rc - \vec{r}\cdot\vec{v}}$$

$$\vec{\nabla}\varphi = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(rc-\vec{r}\cdot\vec{v})^3}\left[(rc-\vec{r}\cdot\vec{v})\vec{v} - (c^2-v^2+\vec{r}\cdot\vec{a})\vec{r}\right]$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{qc}{4\pi\varepsilon_0}\frac{1}{(rc-\vec{r}\cdot\vec{v})^3}\left[(rc-\vec{r}\cdot\vec{v})\left(-\vec{v}+\frac{r\vec{a}}{c}\right) + \frac{r}{c}(c^2-v^2-\vec{r}\cdot\vec{a})\vec{v}\right]$$

$$\vec{E} = -\left(\vec{\nabla}\varphi + \frac{\partial\vec{A}}{\partial t}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r}{(\vec{r}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{r}\times(\vec{u}\times\vec{a})\right]$$

$\vec{u} = \hat{r}c - \vec{v}$

$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\hat{r}}{c}\times\vec{E}$, For a point charge $\vec{E}\perp\vec{B}$

and $\vec{B}\perp\vec{r}$

For $v \ll c$ $\quad \vec{E} = \frac{q}{4\pi\varepsilon}\frac{\hat{r}}{r^2}$