$$\vec{D} = \vec{P} + \varepsilon_0 \vec{E}$$

$$\vec{\nabla}\cdot\vec{D} = \rho_f \; ; \; \vec{\nabla}\cdot\vec{E} = \frac{\rho_{tot}}{\varepsilon_0} \; ; \; \vec{\nabla}\cdot\vec{P} = -\rho_b$$

$$\vec{\nabla}\times\vec{E} = 0 \; ; \; \vec{\nabla}\times\vec{D} = -\vec{\nabla}\times\vec{P} \neq 0 \text{ in general}$$

Boundary conditions

$$D_\perp^+ - D_\perp^- = \sigma_f \; ; \; E_\perp^+ - E_\perp^- = \frac{\sigma_{tot}}{\varepsilon_0} \; , \; P_\perp^+ - P_\perp^- = -\sigma_b$$

$$\vec{E}_{\parallel}^+ = \vec{E}_{\parallel}^- \; ; \; \vec{D}_{\parallel}^+ - \vec{D}_{\parallel}^- = \vec{P}_{\parallel}^+ - \vec{P}_{\parallel}^-$$

Linear dielectrics

$$\vec{P} = \varepsilon_0 \chi \vec{E} \Rightarrow \vec{D} = \varepsilon \vec{E}$$

$$\varepsilon = \varepsilon_0 (1+\chi)$$

χ - susceptibility

ε - permittivity

Point charge in a dielectric medium

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon}\frac{q}{r} \; ; \; \vec{E} = -\vec{\nabla} V$$

$$\vec{\nabla}\cdot\vec{D} = q\,\delta(\vec{r}) \; ; \; \vec{\nabla}\times\vec{D} = 0 \; ; \; \vec{D} = \varepsilon\vec{E}$$

$$\vec{D} = -\vec{\nabla}\varphi \; ; \; \varphi = \frac{1}{4\pi}\frac{q}{r}$$

Non-uniform linear dielectrics

$$D_i = \varepsilon_{ij} E_j$$

$$P_i = \varepsilon_0 \chi_{ij} E_j$$

$$\varepsilon_{ij} = \varepsilon_0 (\delta_{ij} + \chi_{ij})$$

Electrostatic energy in a dielectric medium

$\rho(\vec{r})$ and $\varphi(\vec{r})$

$$\mathcal{E} = \frac{1}{2} \int_\Omega \rho(\vec{r})\, \varphi(\vec{r})\, d\vec{r}$$

Assume $\rho(\vec{r}) \to \rho(\vec{r}) + \delta\rho(\vec{r})$

$$\delta\mathcal{E} = \int_\Omega \delta\rho(\vec{r})\, \varphi(\vec{r})\, d\vec{r}$$

Add extra charge at the position $\vec{r}$ first, change in energy $\varphi(\vec{r})\delta\rho(\vec{r})$.
Then add extra charge at r', change in energy $\varphi(\vec{r}\,')\delta\rho(\vec{r}\,')$. We ignore the change in $\varphi(\vec{r}\,')$, $\delta\varphi(\vec{r}\,')$ due to $\delta\rho(\vec{r})$ since this contributes to the energy only to second order $\delta\varphi(\vec{r}\,')\delta\rho(\vec{r}\,')$

$$\vec{\nabla}\cdot(\delta\vec{D}) = \delta\rho$$

only free charge changes

$$\delta\mathcal{E} = \int_\Omega \vec{\nabla}\cdot(\delta\vec{D})\,\phi(\vec{r})\,d\vec{r}$$

$$= \int_\Omega \delta\vec{D}\cdot\vec{E}\,d\vec{r}$$ (fields go to zero at ∞)

$$\mathcal{E} = \int d\vec{r}\int_0^{\vec{D}} \vec{E}\cdot\delta\vec{D}$$

Linear dielectrics

$$\delta\vec{D}\cdot\vec{E} = \frac{1}{2}\delta(\vec{D}\cdot\vec{E})$$

$$\mathcal{E} = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,d\vec{r}$$

Energy to polarize a dielectric

Initially $\vec{D}_0$ & $\vec{E}_0$

$$\mathcal{E}_0 = \frac{1}{2}\int \vec{E}_0\cdot\vec{D}_0\,d\vec{r}$$

After adding dielectric

$\vec{E}$ & $\vec{D}$

$$\Delta \mathcal{E} = \frac{1}{2} \int \left(\vec{E} \cdot \vec{D} - \vec{E}_0 \cdot \vec{D}_0 \right) d\vec{r}$$

$$= \frac{1}{2} \int \left(\vec{E} \cdot \vec{D}_0 - \vec{D} \cdot \vec{E}_0 \right) d\vec{r}$$

$$+ \frac{1}{2} \int \left(\vec{E} + \vec{E}_0 \right) \cdot \left(\vec{D} - \vec{D}_0 \right) d\vec{r}$$

$$\vec{\nabla} \times \left(\vec{E} + \vec{E}_0 \right) = 0$$

$$\vec{E} + \vec{E}_0 = - \vec{\nabla} \phi$$

$$\int \left(\vec{E} + \vec{E}_0 \right) \cdot \left(\vec{D} - \vec{D}_0 \right) d\vec{r}$$

$$= \int \phi \, \vec{\nabla} \cdot \left(\vec{D} - \vec{D}_0 \right) d\vec{r} = 0$$

$\vec{\nabla} \cdot \left(\vec{D} - \vec{D}_0 \right) = 0$ (free charge density is the same)

$$\vec{D}_0 = \varepsilon_0 \vec{E}_0 \; ; \; \vec{D} = \vec{P} + \varepsilon_0 \vec{E}$$

$$\omega = -\frac{1}{2} \int_\Omega \vec{P} \cdot \vec{E}_0 \, d\vec{r}$$

Ω ; volume where dielectric is