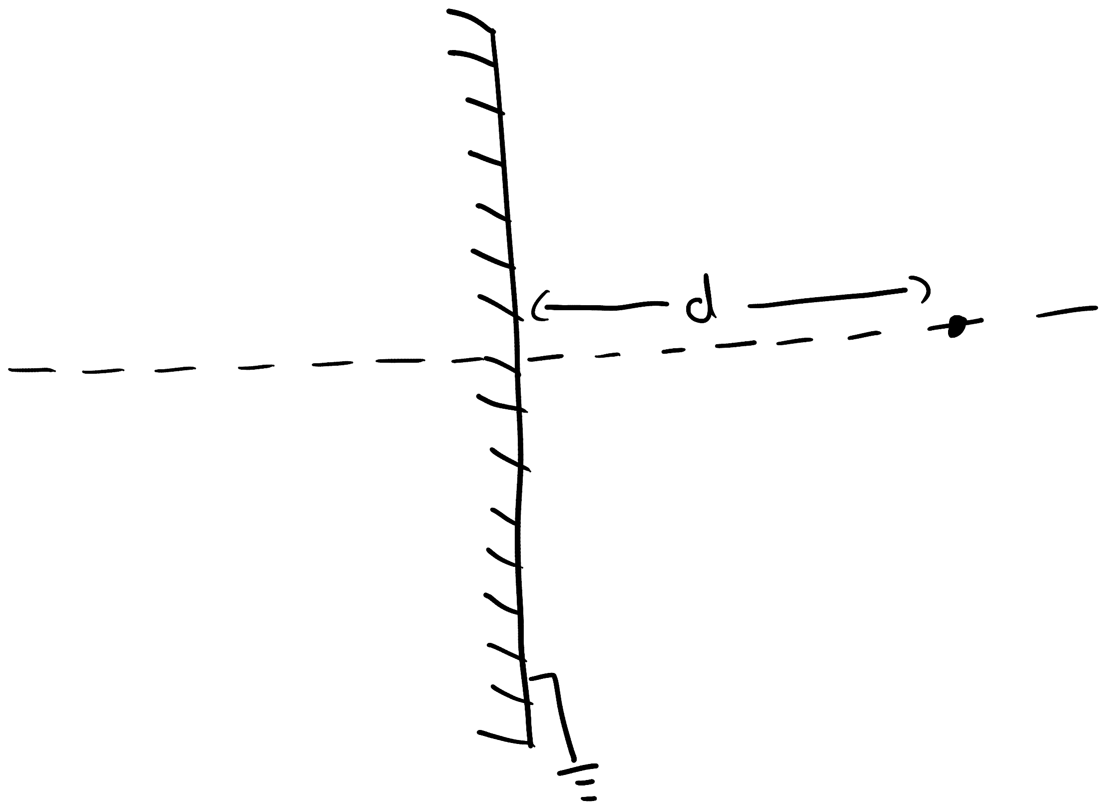
METHOD OF IMAGES

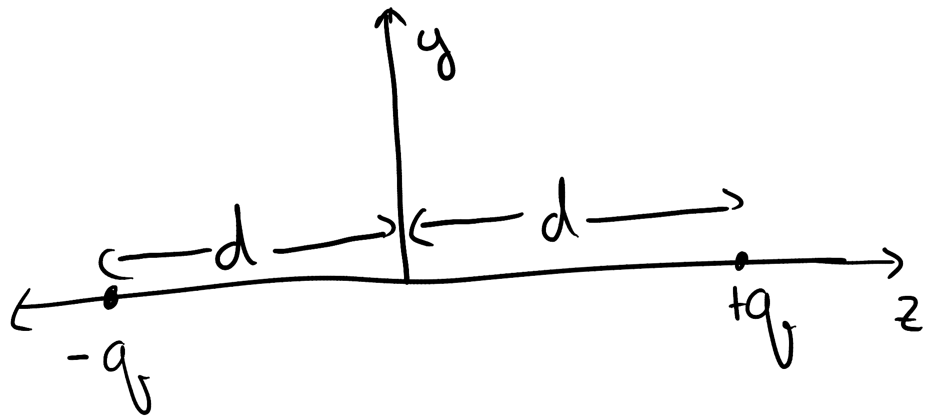


$$\nabla^2 V = 0 \qquad z < 0$$

$$\nabla^2 V = -\frac{q}{\varepsilon_0}\delta(z-d)\delta(x)\delta(y)$$

$$V = 0 \, ; \; z = 0$$

Consider two point charges $+q$ and $-q$ separated by $2d$



$$V = \left[\frac{-q}{\sqrt{y^2+(z+d)^2}} + \frac{q}{\sqrt{y^2+(z-d)^2}}\right]\frac{1}{4\pi\varepsilon_0}$$

$V=0 \implies$

$\implies (z+d)^2 = (z-d)^2$

$\implies z=0$ (zero potential surface)

From rotational symmetry, entire xy plane - zero potential surface.

Original problem solution

$V=0$ for $z<0$

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \quad (z>0)$$ 3

$\vec{E} = 0$ for $z<0$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{x}+y\hat{y}+(z-d)\hat{z}}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{x\hat{x}+y\hat{y}+(z+d)\hat{z}}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right] \quad z>0$$

$\vec{E} = 0 \quad z \to 0^-$

$$= -\frac{q}{4\pi\varepsilon_0}\left[\frac{2d\,\hat{z}}{\left[x^2+y^2+d^2\right]^{3/2}}\right] \quad z \to 0^+$$

$$\sigma(x,y) = \varepsilon_0\left[E_z(z\to 0^+) - E_z(z\to 0^-)\right]$$

$$= -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}}$$

Total induced charge

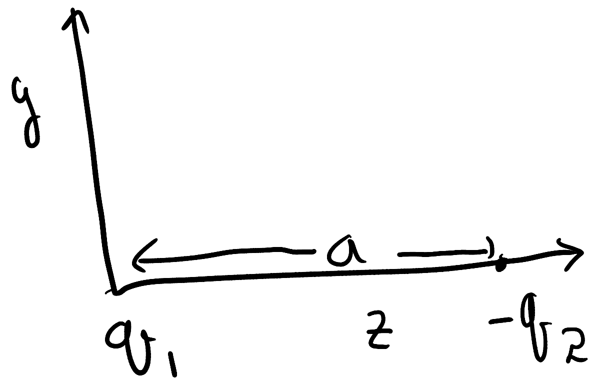
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(x,y)\, dx\, dy$$

$$= -\frac{q}{2\pi} \cdot 2\pi d \int_0^{\infty} \frac{r\, dr}{(r^2+d^2)^{3/2}}$$

$$= -\frac{q d}{2} \left[\frac{-2}{(r^2+d^2)^{1/2}} \Big|_0^{\infty} \right]$$

$$= -q$$

Now consider $+q_1$ and $-q_2$ separated by a distance a



$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{\sqrt{y^2+z^2}} - \frac{q_2}{\sqrt{y^2+(z-a)^2}}\right]$$ ⑤

$$V=0 \Rightarrow \left(\frac{q_1}{q_2}\right)^2\left[y^2+(z-a)^2\right] = y^2+z^2$$

Let $q_1 > q_2$

$$\left[\left(\frac{q_1}{q_2}\right)^2 - 1\right]\left[y^2 + \left(z - \frac{\left(\frac{q_1}{q_2}\right)^2 a}{\left(\frac{q_1}{q_2}\right)^2 - 1}\right)^2\right]$$

$$- \frac{\left(\frac{q_1}{q_2}\right)^2 a}{\left(\frac{q_1}{q_2}\right)^2 - 1} = 0$$

$$y^2 + (z - z_0)^2 = R^2$$

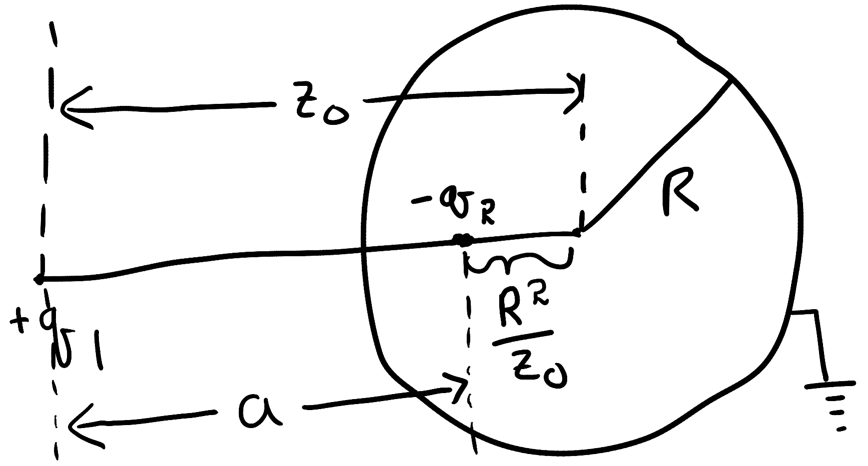
$$z_0 = \frac{\left(\frac{q_1}{q_2}\right)^2 a}{\left(\frac{q_1}{q_2}\right)^2 - 1} \quad ; \quad R = \frac{\frac{q_1}{q_2} a}{\left(\frac{q_1}{q_2}\right)^2 - 1}$$

Symmetry gives circle $\rightarrow$ sphere centred at $(0, 0, z_0)$ of radius R.

$$\frac{R}{z_0} = \frac{q_2}{q_1} < 1$$

$$z_0 > a \quad ; \quad z_0 - R > z_0 - a$$

$$z_0 - a = \frac{R^2}{z_0}$$



Grounded conducting sphere of radius R with charge $+q$ at a distance z_0 from the centre.

Image charge $q' = -\frac{R}{z_0} q$ at a distance $\frac{R^2}{z_0}$ from the centre along the line joining it to and toward q.

Total induced charge is q'.