Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

The last equation $\Rightarrow$ $\vec{\nabla} \cdot \vec{J} = 0$

In reality $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

continuity equation

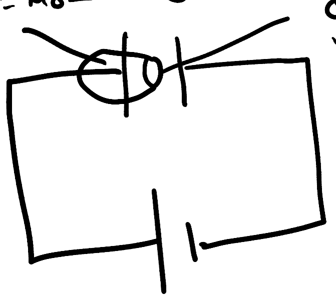
or $\vec{\nabla} \cdot \left[\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] = 0$

Maxwell said

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$\vec{J}_{disp} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ - displacement current

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$ $\qquad$ $\oint \vec{B} \cdot d\vec{l} = 0$



$\vec{J} = \vec{J}_{tot}$

$\rho = \rho_{tot} = \rho_f + \rho_b$

Continuity equations

$$\vec{\nabla} \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J}_p = \vec{\nabla} \cdot \left(\frac{\partial \vec{P}}{\partial t} \right) \quad \text{or} \quad \vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

$$\vec{J}_{tot} = \vec{J}_f + \vec{J}_p + \vec{J}_m$$

$$\vec{\nabla} \times \vec{m} = \vec{J}_m$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{J}_p + \vec{J}_m + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \mu_0 \vec{J}_f + \mu_0 \vec{\nabla} \times \vec{m} + \mu_0 \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's equations in matter

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J}_{disp} = \frac{\partial \vec{D}}{\partial t}$$

Boundary conditions

$$D_\perp^{above} - D_\perp^{below} = \sigma_f$$

$$B_\perp^{above} - B_\perp^{below} = 0$$

$$\vec{E}_{\parallel}^{above} - \vec{E}_{\parallel}^{below} = 0$$

$$\vec{H}_{\parallel}^{above} - \vec{H}_{\parallel}^{below} = \vec{K}_f \times \hat{n}$$

$$E_\perp^{above} - E_\perp^{below} = \sigma_{tot}$$

$$\vec{B}_{\parallel}^{above} - B_\perp^{below} = \mu_0 (\vec{K}_{tot} \times \hat{n})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

or $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \varphi$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{A} \rightarrow \vec{A} + \vec{\nabla} f \Rightarrow \vec{B}$ is invariant

$\vec{E} \rightarrow \vec{E} - \vec{\nabla}\left(\frac{\partial f}{\partial t}\right)$ unless

$\varphi \rightarrow \varphi + \frac{\partial f}{\partial t}$, in which case

$\vec{E} \rightarrow \vec{E}$

Generalized gauge invariance

$$\vec{A} \to \vec{A} + \vec{\nabla} f$$

$$\varphi \to \varphi + \frac{\partial f}{\partial t}$$

Conservation of energy

Configuration of charges

Electrostatic energy

$$W_E = \frac{\varepsilon_0}{2} \int \vec{E}^2 \, d\vec{r}$$

Magnetic energy

$$W_m = \frac{1}{2\mu_0} \int \vec{B}^2 \, d\vec{r}$$

Total electromagnetic energy

$$U_{em} = \frac{\varepsilon_0}{2} \int \vec{E}^2 \, d\vec{r} + \frac{1}{2\mu_0} \int \vec{B}^2 \, d\vec{r}$$

$$\vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l} = q \vec{E} \cdot d\vec{l}$$

$$\frac{dW}{dt} = \int (\vec{E} \cdot \vec{J}) \, d\vec{r}$$

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$\vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}\left[\frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2\right] - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$\frac{dW}{dt} = -\frac{d}{dt}\left[\frac{\varepsilon_0}{2}\int\vec{E}^2\,d\vec{r} + \frac{1}{2\mu_0}\int\vec{B}^2\,d\vec{r}\right]$$

$$\boxed{-\frac{1}{\mu_0}\int\vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\vec{r}}$$

$$\downarrow$$

$$-\frac{1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{a}$$

Poynting's theorem

$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$ – Poynting vector

$$\frac{dW}{dt} = -\frac{dU}{dt} - \oint\vec{S}\cdot d\vec{a}$$