Magnetostatics

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{A}$ - magnetic vector potential

analogous to φ where $\vec{E} = -\vec{\nabla}\varphi$

$\vec{\nabla} \cdot \vec{A}$ has to be specified

Further $\vec{A} \to \vec{A} + \vec{\nabla}\lambda$ gives same

$\vec{B}$ - gauge invariance.

$\vec{\nabla} \cdot \vec{A}$ can always be chosen to be

zero.

Proof: Suppose $\vec{\nabla} \cdot \vec{A} = \vec{V} \neq 0$

Define $\vec{\nabla}\lambda = -\vec{A}$

$\Rightarrow \nabla^2 \lambda = -\vec{\nabla} \cdot \vec{A}$

Solve Laplace's equation for λ

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ has $\vec{\nabla} \cdot \vec{A}' = 0$

Helmholtz Theorem

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

Verify that $\vec{\nabla} \cdot \vec{A} = 0$

$$\vec{B} = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\vec{J}(\vec{r}) \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\vec{r}'$$

Biot - Savart law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad \text{(Ampere's law)}$$

Solenoid

$$\vec{B} = \mu_0 n I \hat{z}$$

If in some volume $\vec{J} = 0$

$\vec{\nabla} \times \vec{B} = 0$ & $\vec{\nabla} \cdot \vec{B} = 0$

$\vec{B} = -\vec{\nabla} \Phi_m$ (like in electrostatics)

and $\nabla^2 \Phi_m = 0$ (Laplace's equation)

Multipole expansion

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int_{\Omega} r'^n P_n(\cos\theta') \vec{J}(\vec{r}') d\vec{r}'$$

Let $\vec{J}(\vec{r}')$ be localized

$$\int_{\Omega} \vec{\nabla}\cdot(\vec{r}'\vec{J})\, d\vec{r}' = \int_{\partial\Omega} (\vec{r}'\vec{J})\cdot d\vec{S}' = 0$$

$$\Rightarrow \int_{\Omega} \vec{J}\, d\vec{r}' = 0 \quad \because \vec{\nabla}'\cdot\vec{J} = 0$$

$\boxed{\text{No monopole term}}$

Dipole term

$$\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^2} \int_{\Omega} r'\cos\theta' \, \vec{J}(\vec{r}') d\vec{r}'$$

$$= \frac{\mu_0}{4\pi r^3} \int_{\Omega} (\vec{r}\cdot\vec{r}')\, \vec{J}(\vec{r}')\, d\vec{r}'$$

$$= \frac{\mu_0}{4\pi r^3} \left[-\vec{r} \times \int_{\Omega} \vec{r}' \times \vec{J}(\vec{r}')\, d\vec{r}' + \int_{\Omega} \vec{r}'(\vec{r}\cdot\vec{J})\, d\vec{r}' \right]$$

Exercise

Show that $\int_{\Omega} \vec{r}'(\vec{r}.\vec{J})d\vec{r}' = -\int(\vec{r}.\vec{r}')\vec{J}(\vec{r}')d\vec{r}'$

$$\vec{A}_p(\vec{r}) = \frac{\mu_0}{4\pi r^3}\vec{m}\times\vec{r}$$

$$\vec{m} = \frac{1}{2}\int_{\Omega} \vec{r}'\times\vec{J}(\vec{r}')d\vec{r}'$$

magnetic dipole moment

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{r}(\hat{r}.\vec{m})-\vec{m}}{r^3}\right]$$

Magnetic field of a dipole

Current in a loop

$$\vec{m} = \frac{I}{2}\oint\vec{r}\times d\vec{l} = I\vec{A}$$

Circular loop $\vec{m} = \pi r^2 I\hat{a}$

$$\vec{J} = \sum_i q_i v_i \delta(\vec{r}-\vec{r}_i)$$

$$\vec{m} = \frac{1}{2}\sum_i q_i \vec{r}_i\times\vec{v}_i = \sum_i \frac{q_i}{2m_i}\vec{L}_i$$

m_i - mass