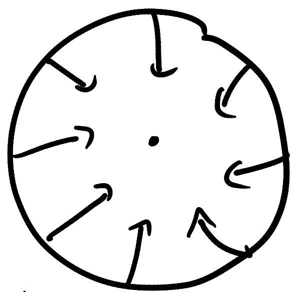
Laplace's equation

$$\nabla^2 V = 0$$

Solutions have no maxima or minima

Earnshaw's theory

Local maximum



$$\vec{E} = -\vec{\nabla} V$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{S} \neq 0 \text{ contradiction}$$

since $\rho = 0$

1D Laplace equation

$$\frac{d^2 V}{dx^2} = 0 \text{ ; solution } V = Ax + B$$

Second order ODE ; BC V at two values or V and $\frac{dV}{dx}$.

In 2D

$$\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) = X(x)\,Y(y)$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = k^2 = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

$$X(x) = A e^{kx} + B e^{-kx} \quad ; \; k \neq 0$$

$$Y(y) = C \sin ky + D \cos ky$$

$$X(x) = mx + p \quad ; \; k = 0$$

$$Y(y) = ry + s$$

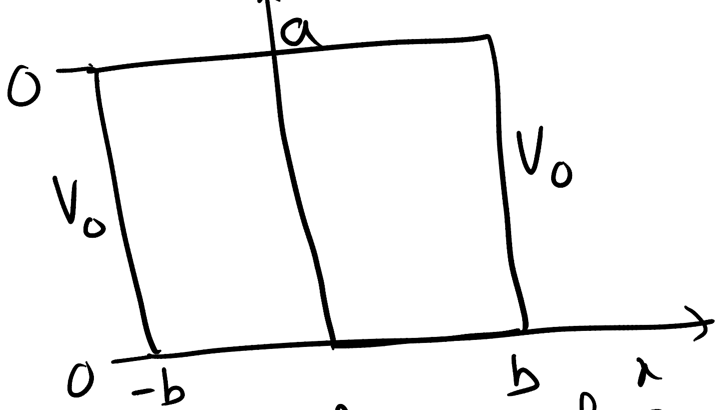
$$V(x,y) = \sum_{k \neq 0} \left[\alpha_k e^{kx} \cos ky + \beta_k e^{-kx} \cos ky + \gamma_k e^{kx} \sin ky + \delta_k e^{-kx} \sin ky \right]$$

$$+ \varepsilon xy + \eta x + \rho y + \xi$$

Consider

$$V(x,y) = 0 \quad \text{for } y=0 \text{ \& } y=a$$

$$V(x,y) = V_0 \quad \text{for } x=-b \text{ \& } x=b$$



$$V(x,0) = \sum_{k\neq 0}\left[\alpha_k e^{kx} + \beta_k e^{-kx}\right] + \eta x + \xi = 0 \quad \forall x \in [-b,b]$$

$$\Rightarrow \alpha_k = \beta_k = \eta = \xi = 0$$

$$V(x,y) = \sum_{k\neq 0}\left(\gamma_k e^{kx} + \delta_k e^{-kx}\right)\sin ky + \varepsilon xy + \rho y$$

$$V(x,a) = \sum_{k\neq 0}\left(\gamma_k e^{kx} + \delta_k e^{-kx}\right)\sin ka + \varepsilon a x + \rho a = 0 \quad \forall x$$

$$\Rightarrow \varepsilon = \rho = 0 \text{ and } \sin ka = 0$$

$$\Rightarrow k = \frac{n\pi}{a}$$

$$V(x,y) = \sum_{n=1}^{\infty} \left(\gamma_n e^{n\pi x/a} + \delta_n e^{-n\pi x/a}\right) \sin\frac{n\pi y}{a}$$ ④

$$V(b,y) = \sum_{n=1}^{\infty} \left(\gamma_n e^{n\pi b/a} + \delta_n e^{-n\pi b/a}\right) \sin\frac{n\pi y}{a}$$

$$= V_0 \quad \forall y$$

$$V(-b,y) = \sum_{n=1}^{\infty} \left(\gamma_n e^{-n\pi b/a} + \delta_n e^{n\pi b/a}\right) \sin\frac{n\pi y}{a}$$

Consider

$$f(y) = \sum_{n=1}^{\infty} \lambda_n \sin\frac{n\pi y}{a}$$

$$\int_0^a \sin\frac{n\pi y}{a} \sin\frac{m\pi y}{a}\, dy = \frac{a}{2}\delta_{mn}$$

$$\Rightarrow \lambda_n = \frac{2}{a}\int_0^a f(y) \sin\frac{n\pi y}{a}\, dy$$

Fourier analysis

Further $f(y) = 0 \Rightarrow \lambda_n = 0$

Thus $\gamma_n = \delta_n$

$$V(x,y) = \sum_{n=1}^{\infty} 2\gamma_n \cosh\frac{n\pi x}{a} \sin\frac{n\pi y}{a}$$

$$V(b,y) = \sum_{n=1}^{\infty} 2\gamma_n \cosh\frac{n\pi b}{a} \sin\frac{n\pi y}{a}$$

$$= V_0$$

$$\Rightarrow \quad 2\gamma_n = 0 \text{ if } n \text{ is even}$$

$$= \frac{4V_0}{n\pi} \text{ if } n \text{ is odd}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cosh\frac{n\pi x}{a} \sin\frac{n\pi y}{a}$$