# Laplace's Equation in 3D

## Cartesian coordinates

$$\nabla^2 V = 0$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V = X(x)\,Y(y)\,Z(z)$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

$$\frac{d^2 Z}{dz^2} = -l^2 Z$$

$$\frac{d^2 Y}{dy^2} = -k^2 Y$$

$$\frac{d^2 X}{dx^2} = (k^2 + l^2) X$$

$$V(x,y,z) = \sum_{k,l}\Big[\left(A e^{\sqrt{k^2+l^2}\,x} + B e^{-\sqrt{k^2+l^2}\,x}\right)\left(C \sin ky + D \cos ky\right)\left(E \sin lz + F \cos lz\right)\Big] + \alpha xyz + \beta xy + \gamma yz + \delta xz + \varepsilon x + \eta y + \zeta z + K$$

# Laplace's equation in spherical polar coordinates

$$\nabla^2 V = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Azimuthal symmetry $\Rightarrow V(r,\theta)$

so $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = m^2 \; ; \; \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -m^2$$

$$R = Ar^l \Rightarrow l(l+1) = m^2$$

$$R = Ar^l + \frac{B}{r^{l+1}}$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + l(l+1)\sin\theta\,\Theta = 0$$

$\cos\theta = x$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + l(l+1)\Theta = 0$$

$$=(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + l(l+1)\Theta = 0$$

Legendre polynomials

$\Theta(x) = P_l(x)$ - $l$-integer.

Let $\Theta(x) = \sum_n a_n x^n$

$$\Rightarrow (1-x^2)\sum_n n(n-1)a_n x^{n-2} - 2x\sum_n n a_n x^{n-1} + l(l+1)\sum_n a_n x^n = 0$$

or $$\sum_n \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + l(l+1)a_n\right]x^n = 0$$

or $$\sum_n \left[(n+2)(n+1)a_{n+2} - \{n(n+1) - l(l+1)\}a_n\right]x^n = 0$$

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} a_n$$

$a_n = 0$ for $n<0$, solution singular at $x=0$.

If $l$ is a non-negative integer
Two solutions, polynomial
of degree $l$ and infinite series.
If $l$ even, infinite series has only
odd terms and vice-versa. Infinite
series solutions are singular at
$x = 1$ or $-1$ and hence discarded.

If $l$ is not an integer, only infinite
series solutions.

Polynomials - Legendre polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = (3x^2 - 1)/2$

$P_3(x) = (5x^3 - 3x)/2$

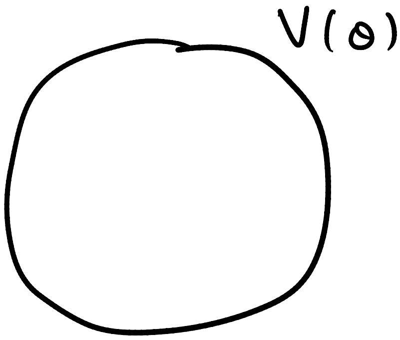
$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

Rodrigues formula

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$ → linearly idpt.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Consider sphere with surface potential




$r < R$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$ ($B_\ell = 0$ otherwise $V(r,\theta)$ would blow up at $r=0$)

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$ ($A_\ell = 0$ or $V$ would blow up at $r=\infty$)

$r > R$

Solution has to be continuous

at $r = R$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$$\Rightarrow A_l = \frac{B_l}{R^{2l+1}}$$

Further

$$V(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

gives $$B_l = \frac{2l+1}{2} R^{l+1} \int_{\theta=0}^{\pi} V(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$