Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Statics

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = 0 \text{ and } \frac{\partial \vec{E}}{\partial t} = 0$$

Electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \; ; \; \vec{\nabla} \times \vec{E} = 0$$

Magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0 \; ; \; \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Consider a vector field $\vec{F}$

$$\vec{\nabla}\cdot\vec{F} = D(\vec{r})$$ in some region Ω

$$\vec{\nabla}\times\vec{F} = \vec{C}(\vec{r})$$

Is $\vec{F}$ uniquely determined?

No! Need boundary conditions.

Let $\vec{F}_1$ and $\vec{F}_2$ be two solutions

and $\vec{\mathcal{F}} = \vec{F}_1 - \vec{F}_2$.

$$\vec{\nabla}\cdot\vec{\mathcal{F}} = 0 \quad \& \quad \vec{\nabla}\times\vec{\mathcal{F}} = 0 \text{ in } \Omega$$

$$\vec{\mathcal{F}} = -\vec{\nabla}\upsilon \quad \because \vec{\nabla}\times\vec{\mathcal{F}} = 0$$

$$\Rightarrow \nabla^2\upsilon = 0 \quad \text{(Laplace's equation)}$$

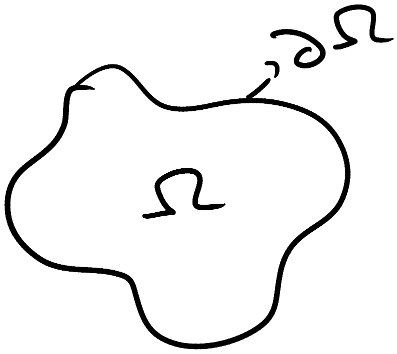
Consider $\int_\Omega (\vec{\nabla}\upsilon)^2 d\vec{r}$

$$= \int_\Omega [\vec{\nabla}\cdot(\upsilon\vec{\nabla}\upsilon) - \upsilon\nabla^2\upsilon]\, d\vec{r}$$

$$= \int_\Omega \vec{\nabla}\cdot(\upsilon\vec{\nabla}\upsilon)\, d\vec{r}$$

$(\nabla^2\upsilon = 0)$

$$\int_{\Omega} \vec{\nabla}\cdot(v\,\vec{\nabla} v)\,d\vec{r} = \int_{\partial\Omega} (v\,\vec{\nabla} v)\cdot d\vec{S}$$



∂Ω, area enclosing Ω

$$\int_{\partial\Omega} (v\,\vec{\nabla} v)\cdot d\vec{S} = -\int_{\partial\Omega} v\,\vec{\tilde{F}}\cdot d\vec{S}$$

If $F_{\partial\Omega}$ is specified (boundary condition); $\vec{\tilde{F}}\cdot d\vec{S} = 0$

$\omega \quad \int_{\Omega} (\vec{\nabla} v)^2\, d\vec{r} = 0$

$\Rightarrow \vec{\nabla} v = \vec{\tilde{F}} = 0$ everywhere in Ω

$\Rightarrow \vec{F}_1 = \vec{F}_2$ and the solution is unique.

Special case $\vec{C} = 0$ (electrostatics)

$$\vec{\nabla} \times \vec{E} = 0 \ ; \ \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$\vec{E} = -\vec{\nabla} V$ (V- electrostatic potential)

$\Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (Poisson's equation)

V_1 and V_2 are two solutions

$\vartheta = V_1 - V_2$; $\vec{E}_1 = -\vec{\nabla} V_1$ & $\vec{E}_2 = -\vec{\nabla} V_2$

$$\int_{\Omega} (\vec{\nabla} \vartheta)^2 \, d\vec{r} = -\int_{\partial\Omega} \vartheta (\nabla \vec{\vartheta}) \cdot d\vec{S}$$

$= 0$ if V is specified on $\partial\Omega$

or $E_{\partial\Omega}$ is specified.

Uniqueness theorem of electrostatics

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ in volume Ω has unique solution provided V is specified on $\partial\Omega$ or $E_{\partial\Omega} = -\nabla_{\partial\Omega} V$ is specified on $\partial\Omega$.

Special subcase.

$\partial\Omega$ is the surface of a conductor

V is a constant on $\partial\Omega$.

V_1 and V_2 are also constant as

is $v = V_1 - V_2$

Thus $$-\int_{\partial\Omega} v(\vec{\nabla} v)\cdot d\vec{S} = -v\int_{\partial\Omega}(\vec{\nabla} v)\cdot d\vec{S}$$

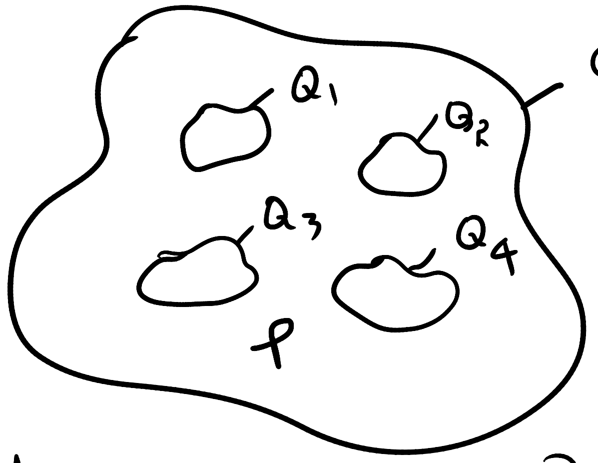
$$= -v\left[\int_{\partial\Omega}\vec{E}_1\cdot d\vec{S} - \int_{\partial\Omega}\vec{E}_2\cdot d\vec{S}\right]$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \int_{\partial\Omega}\vec{E}\cdot d\vec{S} = \frac{Q_{tot}}{\varepsilon_0}$$

Thus $$\int_{\partial\Omega}\vec{E}_1\cdot d\vec{S} - \int_{\partial\Omega}\vec{E}_2\cdot d\vec{S}$$

$$= \frac{Q_{1,tot}}{\varepsilon_0} - \frac{Q_{2,tot}}{\varepsilon_0} = 0$$

if Q_{tot} is specified.



Specify Q_1, Q_2, Q_3, Q_4 etc.

$\vec{E}$ is uniquely determined

Helmholtz Theorem

$$\vec{\nabla}\cdot\vec{F} = D(\vec{r})$$

$$\vec{\nabla}\times\vec{F} = \vec{C}(\vec{r})$$

and appropriate boundary conditions for uniqueness, how do you determine $\vec{F}$?

Let boundary be at ∞.

$$\vec{\nabla}\cdot\vec{C} = 0 \quad (\because \vec{\nabla}\times\vec{F} = \vec{C})$$

$$\vec{F} = -\vec{\nabla} u + \vec{\nabla} \times \vec{\omega}$$

$$u(\vec{r}) = \frac{1}{4\pi} \int \frac{D(\vec{r}')\, d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\vec{\omega}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')\, d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \cdot \vec{F} = -\nabla^2 u = -\frac{1}{4\pi} \int D(\vec{r}')\, \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d\vec{r}'$$

$$= \frac{1}{4\pi} \int D(\vec{r}')\, 4\pi\, \delta^3(\vec{r} - \vec{r}')\, d\vec{r}'$$

$$= D(\vec{r})$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{\nabla} \times \vec{\omega}) = -\nabla^2 \vec{\omega} + \vec{\nabla}(\vec{\nabla} \cdot \vec{\omega})$$

$-\nabla^2 \vec{\omega} = \vec{C}(\vec{r})$ [Same calculation as $-\nabla^2 u$]

$$\vec{\nabla} \cdot \vec{\omega} = \frac{1}{4\pi} \int \vec{C}(\vec{r}') \cdot \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|}\, d\vec{r}'$$

$$= -\frac{1}{4\pi} \int \vec{C}(\vec{r}') \cdot \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|}\, d\vec{r}'$$

$$= \frac{1}{4\pi} \int \frac{\vec{\nabla}' \cdot \vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d\vec{r}' - \frac{1}{4\pi} \int \vec{\nabla}' \cdot \left(\frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d\vec{r}'$$

$\vec{\nabla}' \cdot \vec{C} = 0$

and $\displaystyle\int \vec{\nabla}' \cdot \left[\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right] d\vec{r}'$

$$= \int_S \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} \cdot d\vec{S}'$$

$$\vec{\nabla} \cdot \vec{\omega} = -\frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} \cdot d\vec{S}' \quad - (1)$$

We want $\vec{\nabla}(\vec{\nabla} \cdot \vec{\omega})$ to be zero.

$u(\vec{r})$ will exist if

$\displaystyle\frac{1}{4\pi} \int \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}'$ is convergent.

At infinity ($\vec{r}' \rightarrow \infty$), the integral will look like

$$\frac{1}{4\pi} \int^{\infty} \frac{D(\vec{r}')}{r'} r'^2 dr'$$

Thus $D(\vec{r}')$ will have to fall off faster than $\frac{1}{r'^2}$.

Similarly $\vec{C}(\vec{r}\,')$ also has to fall off faster than $\frac{1}{r'^2}$.

This also ensures that

$$\vec{\nabla}\cdot\vec{\omega} = -\frac{1}{4\pi}\int\frac{\vec{C}(\vec{r}\,')\cdot d\vec{S}\,'}{|\vec{r}-\vec{r}\,'|} \rightarrow 0$$

Thus

$$\vec{F} = -\vec{\nabla}u + \vec{\nabla}\times\vec{\omega}$$

$$= \frac{1}{4\pi}\int\frac{D(\vec{r}\,')\,(\vec{r}-\vec{r}\,')\,d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \rightarrow \text{Coulomb}$$

$$+ \frac{1}{4\pi}\int\frac{\vec{C}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d\vec{r}\,' \rightarrow \text{Biot-Savart}$$