Green's functions

Poisson's equation

$$\nabla^2 \varphi = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\rho(\vec{r}) = \int_V \rho(\vec{r}')\, \delta(\vec{r}-\vec{r}')\, d\vec{r}'$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int G(\vec{r},\vec{r}')\, \rho(\vec{r}')\, d\vec{r}'$$

$$\nabla^2 G(\vec{r},\vec{r}') = -4\pi\, \delta(\vec{r}-\vec{r}')$$

$G(\vec{r},\vec{r}')$ - Green's function

For matrices

$$A X = Y \; ; \quad X = A^{-1} Y$$

$$G = A^{-1} \qquad GA = AG = I$$

Boundary at ∞ with $\varphi(r \to \infty) \to 0$

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d\vec{r}'$$

What about for some other boundary conditions?

Green's theorem

$$\int_{\Omega} [\varphi \nabla^2 \psi - \psi \nabla^2 \varphi] \, d\vec{r} = \int_{\partial\Omega} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS$$

$$\frac{\partial}{\partial \vec{n}} = \frac{\partial}{\partial n} \hat{n}$$

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$$

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$$

$F(\vec{r}, \vec{r}')$ solution of $\nabla^2 F = 0$ and set by boundary conditions

$$\psi = G(\vec{r} - \vec{r}')$$

$$\int_{\Omega} \varphi \left[-4\pi \delta(\vec{r} - \vec{r}') \right] d\vec{r}' + \int_{\Omega} G(\vec{r} - \vec{r}') \frac{\rho(\vec{r}')}{\varepsilon_0} d\vec{r}' = \int_{\partial\Omega} \left(\varphi \frac{\partial G}{\partial n'} - G \frac{\partial \varphi}{\partial n'} \right) dS'$$

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$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \rho(\vec{r}')\, G(\vec{r}-\vec{r}')\, d\vec{r}'$$

$$+ \frac{1}{4\pi} \int_{\partial\Omega} \left(G \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial G}{\partial n'} \right) dS'$$

Dirichlet conditions, choose F such that

$G = 0$ on $\partial\Omega$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\Omega} \rho(\vec{r}')\, G(\vec{r}-\vec{r}')\, d\vec{r}'$$

$$- \frac{1}{4\pi} \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n'}\, dS'.$$

Neumann conditions

$\frac{\partial G}{\partial n'} = 0$ on $\partial\Omega$ doesn't work

$$\nabla^2 G = -4\pi\delta(\vec{r}-\vec{r}')$$

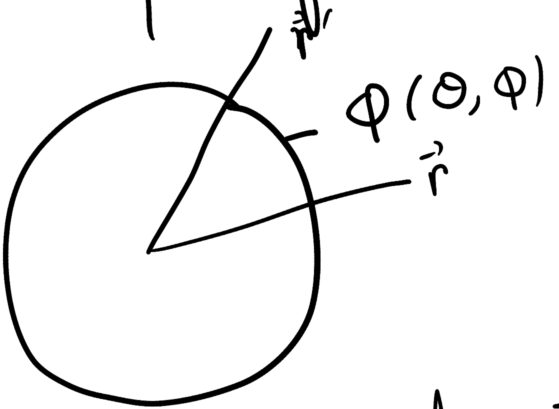
$$\Rightarrow \int_{\partial\Omega} \frac{\partial G}{\partial n'}\, dS' = -4\pi$$

so choose $\frac{\partial G}{\partial n'} = -\frac{4\pi}{S}$; S area of $\partial\Omega$

$$\varphi(\vec{r}) = \langle \varphi \rangle_{\partial\Omega} + \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r}\,')\, G(\vec{r}-\vec{r}\,')\, d\vec{r}\,'$$

$$+ \frac{1}{4\pi}\int_{\partial\Omega} G \frac{\partial \varphi}{\partial n'}\, dS'$$

Example of the use of Green's functions



Dirichlet conditions
$G(\vec{r}, \vec{r}\,') = 0$

Green's function for the sphere

$$G(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a}{r'\left|\vec{r} - \frac{a^2}{r'^2}\vec{r}\,'\right|}$$

$$G(\vec{r}, \vec{r}\,') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma}}$$

$$\cos\gamma = \cos(\theta - \theta') + \sin\theta \sin\theta' \cos(\varphi - \varphi')$$

$$\frac{\partial G}{\partial n'} = -\frac{(r^2 - a^2)}{a(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}$$

Let $\rho = 0$

$$\phi(\vec{r}, \theta, \phi) = \frac{1}{4\pi} \int \frac{\phi_0(a, \theta', \phi')\, a(r^2 - a^2)}{a(r^2 + a^2 - 2ar\cos\gamma)^{3/2}}\, d\Omega'$$