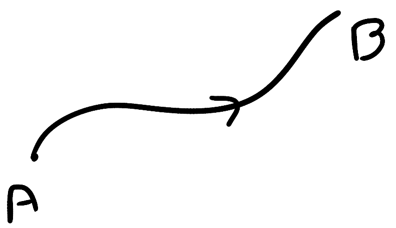
Electrostatics

(no currents flow)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\varphi$$



$$\int_A^B \vec{E}\cdot d\vec{l} = \varphi_A - \varphi_B$$

φ - Electrostatic potential

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\vec{r}\,' \; ; \; \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\, d\vec{r}\,'$$

Note $V(\vec{r}) + C$ also gives same $\vec{E}$.

Fix $V(\vec{r})$ at some point (often $V(\vec{r}) = 0$ at $\vec{r} = \infty$)

to get unique $V(\vec{r})$

$-\nabla^2 V = \frac{\rho}{\varepsilon_0}$ Poisson's equation

If $\rho = 0$

$\nabla^2 V = 0$ - Laplace's equation

$\nabla^2 \longrightarrow$ Laplacian

Uniqueness of V.

Let V_1 and V_2 be two solutions of

$-\nabla^2 V = \frac{\rho}{\varepsilon_0}$; $\upsilon = V_1 - V_2 \Rightarrow \nabla^2 \upsilon = 0$

$$\int_\Omega (\nabla \upsilon)^2 d\vec{r} = \int_\Omega \vec{\nabla}\cdot(\upsilon \vec{\nabla} \upsilon)\, d\vec{r} - \underbrace{\boxed{\int_\Omega \upsilon \nabla^2 \upsilon\, d\vec{r}}}_{=0}$$

$$= \int_{\partial\Omega} \upsilon \vec{\nabla} \upsilon \cdot d\vec{S}$$

$= 0$ if $\upsilon = 0$ or $\vec{\nabla}_n \upsilon = 0$
$\Rightarrow \upsilon = \text{Cons}$

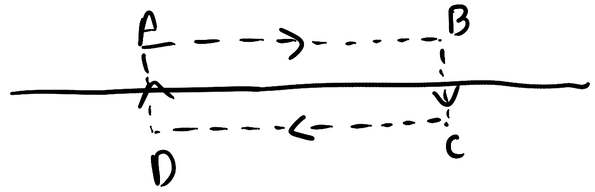
Either specify V on $\partial\Omega$ or $\vec{E}_n$.

V - Dirichlet boundary conditions

$\vec{E}_n = -\frac{\partial V}{\partial n}$ - Neumann boundary conditions.

$\vec{E}$ across boundaries

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$$



$$\oint \vec{E}\cdot d\vec{l} = 0$$

Shrink AD and BC to zero

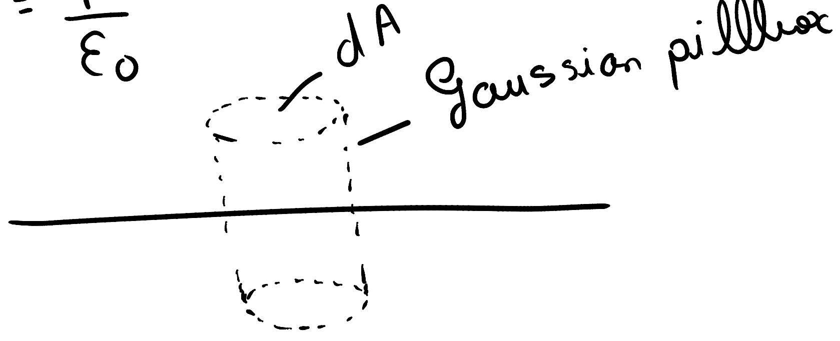
$$\Rightarrow \int_A^B \vec{E}\cdot d\vec{l} = \int_D^C \vec{E}\cdot d\vec{l}$$

Shrink AB and DC to points

$$\Rightarrow \vec{E}_{\parallel}^{\,above} = \vec{E}_{\parallel}^{\,below}$$

$\vec{E}_{\parallel}$ is continuous across a boundary

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$



$$\Rightarrow \oint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

Shrink pillbox to zero volume

$$(\vec{E}\cdot d\vec{A})^{up} + (\vec{E}\cdot d\vec{A})^{down} = \frac{dQ}{\varepsilon_0}$$

$$\Rightarrow E_\perp^{up} - E_\perp^{down} = \frac{\sigma}{\varepsilon_0}$$

σ - surface charge density

$$E_\perp^{up} - E_\perp^{down} = \sigma/\varepsilon_0$$
$$\vec{E}_{\parallel}^{up} = \vec{E}_{\parallel}^{down}$$

Remain the same even in electrodynamics

Conductors

$\vec{E} = 0$ in the bulk

$\Rightarrow \rho = 0$ in the bulk

Also $\vec{E}_{\parallel} = 0$ on the surface

so $\vec{E}$ is always normal to the surface.

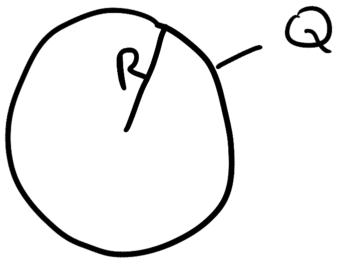
$E_\perp = \sigma/\varepsilon_0$ (surface charge only)

Uniqueness theorem for conductors

Only information required for conductors is total amount of charge on them.

Surface of a conductor - equipotential

Spherical conductor



$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} \quad r > R$$

$$= \frac{Q}{4\pi\varepsilon_0 R} \quad r \leq R$$

V is constant inside a conductor

$\therefore \vec{E} = 0.$

For any isolated conductor

$V(r) \propto Q$.

<u>Proof:</u> The surface charge on a conductor is $\sigma(\vec{r}')$

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int_{\partial\Omega} \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|} dS'$$

satisfies boundary conditions

$V(r) \to 0$ as $r \to \infty$ and

$V(r) =$ cons on $\partial\Omega$.

Further $\int_{\partial\Omega} \sigma(\vec{r}')\, dS' = Q$

Consider a different amount of charge Q' on the conductor.

$$V'(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\frac{Q'}{Q}\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dS'$$

This obeys $V'(\vec{r}) \to 0$ as $\vec{r} \to \infty$

and $V'(\vec{r})$ is a const. on $\partial\Omega$.

Further $\sigma'(\vec{r}\,') = \frac{Q'}{Q}\sigma(\vec{r}\,')$ also

gives $\int_{\partial\Omega} \sigma'(\vec{r}\,')\, dS' = Q'$.

Thus, in general for a total charge Q

$\sigma_Q(\vec{r}) = Q f(\vec{r})$; $f(\vec{r})$ depends

only on geometry.

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\int \frac{f(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, dS'$$

In particular

$$V = \frac{Q}{4\pi\varepsilon_0} \int_{\partial\Omega} \frac{f(\vec{r}\,')}{|\vec{r}_s - \vec{r}\,'|} \, dS'$$

$\vec{r}_s$ is on $\partial\Omega$ and V is the potential

o $\partial\Omega$.

$$Q = CV, \quad C = \frac{4\pi\varepsilon_0}{\int_{\partial\Omega} \frac{f(\vec{r}\,')}{|\vec{r}_s - \vec{r}\,'|} \, dS'} \quad \text{- capacitance}$$

Note C is independent of $\vec{r}_s$ which constrains $f(\vec{r}\,')$

For a spherical conductor

$$C = 4\pi\varepsilon_0 R$$

For conductors Q_i at potentials

$$V_i = \sum_{ij} B_{ij} Q_j$$

Or

$$Q_i = \sum_{ij} C_{ij} V_j$$

C_{ij} - coefficients of capacitance

C_{ii} - self capacitance