EM waves in metals ①

Metals $\vec{J}_f = \sigma \vec{E}$

σ - conductivity

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\varepsilon_0}$$

$$\vec{\nabla} \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$$

$$\vec{J}_f = \sigma \vec{E}$$

$$\sigma \vec{\nabla} \cdot \vec{E} = -\frac{\partial \rho_f}{\partial t}$$

$$\frac{\sigma}{\varepsilon_0} \rho_f = -\frac{\partial \rho_f}{\partial t}$$

$$\rho_f = \rho_0 e^{-\frac{\sigma}{\varepsilon_0} t}$$

$$\rho_f(t \to \infty) = 0$$

So, no free charge in the bulk

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$$

$$k^2 = i\mu\sigma\omega + \frac{\omega^2}{v^2}$$

$$K = K_R + iK_I$$

$$K_R = \frac{\omega}{\sqrt{2}\, v}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$K_I = \frac{\omega}{\sqrt{2}\, v}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

$$\vec{E} = \vec{E}_0 e^{-K_I z} e^{i(K_R z - \omega t)}$$

exponentially decaying over a length

$d = \frac{1}{K_I}$ = skin depth

$\vec{K}\cdot\vec{E} = 0$ from $\vec{\nabla}\cdot\vec{E} = 0$

$\vec{E}$ in the x direction

$$\vec{E} = E_0 e^{-K_I z} e^{i(K_R z - \omega t)}\,\hat{x}$$

$$\Rightarrow \vec{B} = \frac{K}{\omega} E_0 e^{-K_I z} e^{i(K_R z - \omega t)}\,\hat{y}$$

$$K = |K| e^{i\varphi}$$

$$K = \omega \sqrt{\varepsilon \mu \sqrt{1 + \left(\frac{\sigma}{\varepsilon \omega}\right)^2}}$$

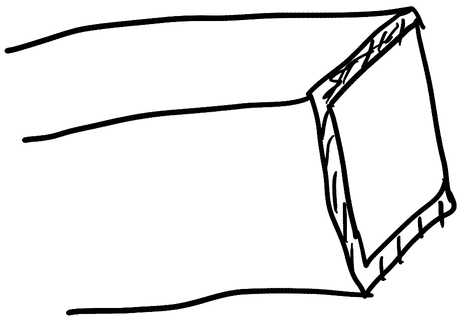
$$\varphi = \tan^{-1}\left(\frac{K_I}{K_R}\right)$$

$$B_0 e^{i\delta_B} = E_0 e^{i\delta_E} \frac{|K| e^{i\varphi}}{\omega}$$

$$\Rightarrow \delta_B - \delta_E = \varphi$$ - phase difference

$$\frac{B_0}{E_0} = \frac{|K|}{\omega}$$

WAVEGUIDES



Waves inside a hollow pipe whose surface is a perfect conductor

$\vec{E} = 0$ & $\vec{B} = 0$

Boundary conditions at the walls

$\vec{E}_{\parallel} = 0$ & $B_{\perp} = 0$

Monochromatic waves propagating along the pipe in the $\hat{z}$ direction

$$\vec{E} = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{B} = \vec{B}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z ; \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y , \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2} E_y$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x ; \quad \frac{\partial B_z}{\partial y} - ikB_y = \frac{i\omega}{c^2} E_x$$

from $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$; $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - K^2}\left[K\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right]$$

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - K^2}\left[K\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right]$$

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - K^2}\left[K\frac{\partial B_z}{\partial x} - \omega\frac{\partial E_z}{\partial y}\right]$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - K^2}\left[K\frac{\partial B_z}{\partial y} + \omega\frac{\partial E_z}{\partial x}\right]$$

$\vec{\nabla}\cdot\vec{E} = 0$ or $\vec{\nabla}\cdot\vec{B} = 0$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - K^2\right]E_x(y,z) = 0$$

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2} + (\omega/c)^2 - K^2\right]B_x(y,z) = 0$$

$E_z = 0 \Rightarrow$ Transverse electric (TE) waves

$B_z = 0 \Rightarrow$ Transverse magnetic (TM) waves

$E_z = B_z = 0 \Rightarrow$ Transverse electromagnetic waves (TEM)

TE waves in a rectangular waveguide

$$B_z(x,y) = X(x)\,Y(y)$$

$$X(x) = C\cos(K_x x) + D\sin(K_x x)$$

At the y boundary ($x=0$ & $x=a$)

$B_y = 0$ & $E_x = 0$

$$\Rightarrow \frac{\partial B_z}{\partial x} = 0$$

$$\Rightarrow K_x = \frac{m\pi}{a}$$

IIIly $K_y = \frac{n\pi}{a}$

$$\left(\frac{\omega}{c}\right)^2 = K_x^2 + K_y^2 + K^2$$

$$\Rightarrow K = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

If $\frac{\omega}{c} < \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

K is imaginary

$\omega_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$ is the cut-off frequency

for the TE_{mn} mode

$\omega_{10} = \frac{c\pi}{a}$ (assuming $a > b$) is the

lowest frequency that can propagate

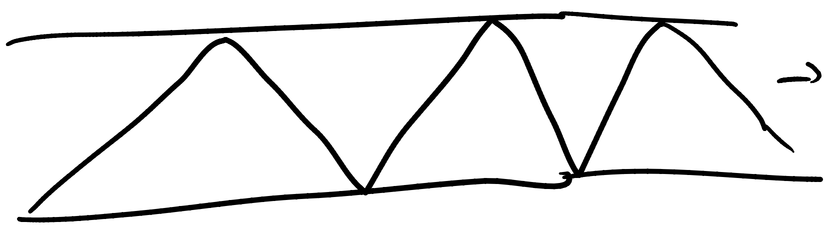
$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} > c \text{ (phase velocity)}$$

but group velocity

$$v_g = \frac{1}{\frac{d\omega}{dk}} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$

which is the speed at which energy

travels



Wave reflected at both interfaces.

Exercise: TM mode

TEM mode impossible in hollow waveguide

Suppose $B_z = 0$ & $E_z = 0$

$$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\vec{E} = E_x(x,y)\hat{x} + E_y(x,y)\hat{y}$$

$\vec{\nabla} \cdot \vec{E} = 0$ & $\vec{\nabla} \times \vec{E} = 0$

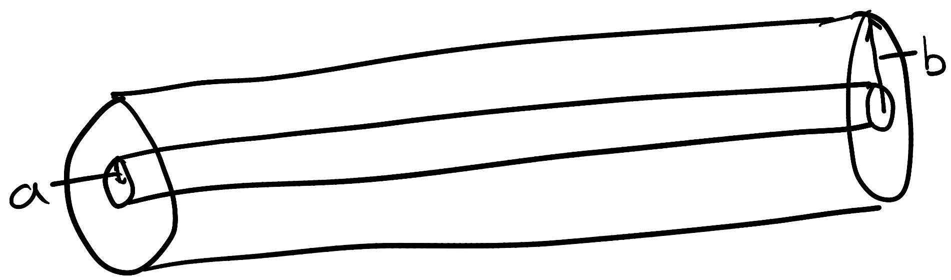
$\Rightarrow \vec{E} = -\vec{\nabla}\varphi$ with $\nabla^2 \varphi = 0$

$\vec{E} = 0$ at the boundary $\because \vec{E} = \vec{E}_{\parallel}$

$\Rightarrow \vec{E} = 0$ everywhere.

Coaxial transmission line

TEM modes are possible



$k = \omega / c$

$c B_y = E_x$ and $c B_x = -E_y$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \; ; \; \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \; ; \; \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

Equations of electrostatics & magnetostatics in 2D with axial symmetry

Infinitely long conductor and infinite solenoid

$$\vec{E}_0(\vec{r}, \phi) = \frac{A}{r}\hat{r} \; ; \; \vec{B}_0(r, \phi) = \frac{A}{cr}\hat{\phi}$$

$$\vec{E}(\vec{r}, \phi, z, t) = \frac{A}{r}\cos(kz - \omega t)\,\hat{r}$$

$$\vec{B}(\vec{r}, \phi, z, t) = \frac{A}{cr}\cos(kz - \omega t)\,\hat{\phi}$$