Conservation laws

$$\frac{dW}{dt} = \frac{d\,U_{mech}}{dt}$$

Work done on charges will increase their mechanical energy KE + PE

Thus $\frac{d}{dt}(U_{mech} + U_{em}) = -\oint\!\!\oint \vec{S}\cdot d\vec{a}$

If the volume is finite in general the total energy is U_{em} + surface terms $\oint\!\!\oint \vec{S}\cdot d\vec{a}$ transports volume energy

If surface is at ∞, surface terms are zero but $\oint\!\!\oint \vec{S}\cdot d\vec{a}$ is not

$$U = \int_\Omega u\, d\vec{r} \;;\; \frac{dU}{dt} = \int_\Omega \frac{\partial u}{\partial t}\, d\vec{r}$$

So $\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$

Force & Momentum

$$\vec{F} = \int \rho(\vec{E} + \vec{v} \times \vec{B})\, d\vec{r}$$

$$\vec{f} = \rho(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$$

$$= \varepsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + \left[\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right]\times\vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$f = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] - \frac{1}{\mu_0}\left[\vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

$$- \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$= \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})\right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

$$- \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$= \varepsilon_0 \left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} \right] + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B} \right] \quad 3$$

$$- \frac{1}{2} \vec{\nabla} \left[\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right] - \varepsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})$$

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

$$f_i = \frac{\partial}{\partial x_j} T_{ij} - \varepsilon_0 \mu_0 \frac{\partial S_i}{\partial t}$$

T_{ij} - Maxwell's Stress Tensor

$$\vec{F} = \int_\Omega \vec{f}\, d\vec{r} = \int_\Omega \vec{\nabla} \cdot \overleftrightarrow{T}\, d\vec{r} - \varepsilon_0 \mu_0 \frac{d}{dt} \int \vec{S}\, d\vec{r}$$

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt}, \quad \frac{d\vec{p}_{tot}}{dt} = \int_\Omega \vec{\nabla} \cdot \overleftrightarrow{T}\, d\vec{r}$$

$$\vec{p}_{em} = \varepsilon_0 \mu_0 \int \vec{S}\, d\vec{r}$$

$\vec{p}_{em} = \varepsilon_0 \mu_0 \vec{S} = \varepsilon_0 (\vec{E} \times \vec{B})$; momentum of EM field

$\dfrac{\partial \vec{p}_{tot}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T}$; $-\overleftrightarrow{T} \rightarrow$ momentum flux density.