Covariant Hamiltonian formalism

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - J^{\alpha} A_{\alpha}$$

For discrete systems

$$H = \sum_i p_i \dot{q}_i - L$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

For continuous systems

$$L = \int \mathcal{L} \, d^3x$$

Momentum density

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}}$$

Hamiltonian density

$$\mathcal{H} = \pi^{\mu} \dot{A}_{\mu} - \mathcal{L}$$

$$\dot{A}_\mu = c\partial_0 A_\mu$$

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \frac{\partial}{\partial \dot{A}_\mu}\left[-\frac{1}{4\mu_0}F^{\alpha\beta}F_{\alpha\beta} - j^\alpha A_\alpha\right]$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

$$= g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\alpha A_\beta \,;\; F_{\gamma\delta} = \partial_\gamma A_\delta - \partial_\delta A_\gamma$$

$$\pi^\mu = -\frac{1}{4\mu_0 c}g^{\alpha\gamma}g^{\beta\delta}\Big[(\delta_{0\gamma}\delta_{\mu\delta} - \delta_{0\delta}\delta_{\mu\gamma})(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$
$$+ (\delta_{0\alpha}\delta_{\mu\beta} - \delta_{0\beta}\delta_{\mu\alpha})(\partial_\gamma A_\delta - \partial_\delta A_\gamma)\Big]$$

$$= -\frac{1}{4\mu_0 c}\Big[(g^{\alpha 0}g^{\beta\mu} - g^{\alpha\mu}g^{\beta 0})(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$
$$+ (g^{0\gamma}g^{\mu\delta} - g^{\mu\gamma}g^{0\delta})(\partial_\gamma A_\delta - \partial_\delta A_\gamma)\Big]$$

$$= -\frac{1}{4\mu_0 c}\Big[\partial^0 A^\mu - \partial^\mu A^0 - \partial^\mu A^0 + \partial^0 A^\mu$$
$$+ \partial^0 A^\mu - \partial^\mu A^0 - \partial^\mu A^0 + \partial^0 A^\mu\Big]$$

$$= -\frac{1}{\mu_0 c}\left[\partial^0 A^\mu - \partial^\mu A^0\right]$$

$$\pi^0 = 0 \; ; \quad \pi^i = -\frac{1}{\mu_0 c^2}\frac{\partial A^i}{\partial t} - \frac{1}{\mu_0 c}\frac{\partial A^0}{\partial x^i} \; ; \; i = 1,2,3$$

$$\pi^0 = 0 \; ; \quad \vec{\pi} = \frac{1}{\mu_0 c^2}\left[-\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}\right] = \frac{\vec{E}}{\mu_0 c^2} = \varepsilon_0 \vec{E}$$

$$\mathcal{H} = \pi^\mu \dot{A}_\mu - \mathcal{L}$$

$$= \pi^\mu \dot{A}_\mu + \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu$$

$$\pi^\mu \dot{A}_\mu = -\vec{\pi} \cdot \frac{\partial \vec{A}}{\partial t} = -\frac{\vec{E}}{\mu_0 c} \cdot \frac{\partial \vec{A}}{\partial t}$$

$$= \frac{\vec{E}}{\mu_0 c^2} \cdot (\vec{E} + \vec{\nabla}\varphi) = \frac{E^2}{\mu_0 c^2} + \frac{\vec{E} \cdot \vec{\nabla}\varphi}{\mu_0 c^2}$$

$$= \varepsilon_0 \left[E^2 + \vec{E} \cdot \vec{\nabla}\varphi\right]$$

$$\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2\mu_0} B^2 - \frac{\varepsilon_0}{2} E^2$$

$$j^\mu A_\mu = \rho\varphi - \vec{j} \cdot \vec{A}$$

$$\mathcal{H} = \frac{1}{2}\left[\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right] - \vec{j} \cdot \vec{A} + \varepsilon_0 \vec{\nabla} \cdot (\vec{E}\varphi)$$

$$H = \int \mathcal{H} \, d^3x$$

$$\varepsilon_0 \int \vec{\nabla} \cdot (E\varphi) \, d^3x = 0$$ for localized distributions

So, $\mathcal{H} = \frac{1}{2}\left[\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right] - \vec{j} \cdot \vec{A}$

Plane waves in a box

$\vec{j} = 0$

$$\mathcal{H} = \frac{1}{2}\left[\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right] = \frac{1}{2}\left[\frac{\pi^2}{\varepsilon_0} + \frac{(\vec{\nabla} \times \vec{A})^2}{\mu_0}\right]$$

Plane wave solutions of $\Box^2 \vec{A} = 0$, we take $\varphi = 0$

Define $\vec{a}_{\vec{k}\lambda} = \frac{1}{\sqrt{V}} \vec{\varepsilon}_{\vec{k}\lambda} \, e^{i\vec{k}\cdot\vec{r}}$

$$\vec{A}(\vec{r},t) = \sqrt{\mu_0} \sum_{\vec{k},\lambda} \vec{u}_{\vec{k}\lambda} q_{\vec{k}\lambda}(t)$$

$$\vec{\Pi}(\vec{r},t) = \sqrt{\varepsilon_0} \sum_{\vec{k},\lambda} \vec{u}_{\vec{k}\lambda} p_{\vec{k}\lambda}(t)$$

$$\varphi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{k}\cdot\vec{\varepsilon}_{\vec{k}\lambda} = 0 \quad \text{(Transverse waves)}$$

Two components $\vec{\varepsilon}_{\vec{k}1}$ or $\vec{\varepsilon}_{\vec{k}2}$ for

er $\vec{k}$

$$\vec{\nabla}\times\vec{u}_{\vec{k}1} = ik\,\vec{u}_{\vec{k}2}$$

$$H = \int \mathcal{H}\, d^3x$$

$$\int \pi^2 d^3x = \varepsilon_0 \sum_{\vec{k}} \sum_{\lambda=1}^{2} |p_{\vec{k}\lambda}(t)|^2$$

$\vec{\pi}$ is real $\Rightarrow u_{\vec{k}\lambda} = u^*_{-\vec{k}\lambda}$ a $p_{\vec{k}\lambda} = p^*_{-\vec{k}\lambda}$

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$$\int (\vec{\nabla}\times\vec{A})^2 d^3x = \mu_0 \sum_{\vec{k}} \sum_{\lambda=1}^{2} |ik\, q_{\vec{k}\lambda}|^2$$

$$= \sum_{\vec{k}} \sum_{\lambda=1}^{2} k^2 |q_{\vec{k}\lambda}|^2$$

$$H = \frac{1}{2} \sum_{\vec{k},\lambda} \left[|p_{\vec{k}\lambda}|^2 + k^2 |q_{\vec{k}\lambda}|^2 \right]$$

$$\{ A^i(\vec{r}), \pi^j(\vec{r}\,') \} = \delta_{ij}\, \delta(\vec{r} - \vec{r}\,')$$

$$\Rightarrow \{ q_{\vec{k}\lambda}, p_{\vec{k}'\lambda'} \} = \delta_{\lambda\lambda'}\, \delta(\vec{k} - \vec{k}')$$

The EM field is a sum of harmonic oscillators labelled by $\vec{k}$ and λ and with frequency $\omega = ck$.