Completely covariant Lagrangian (1)

(Space & time on equal footing)

$$\gamma L = -m_0 c^2$$

$$L = -\frac{m_0 c^2}{\gamma} = -\frac{m_0 c \sqrt{u^\alpha u_\alpha}}{\gamma}$$

$$S = \int_{E_1}^{E_2} L\, dt = -m_0 c \int_{E_1}^{E_2} \sqrt{u^\alpha u_\alpha}\, \frac{dt}{\gamma}$$

E_1 & E_2 are space-time ev-s

$$S = -m_0 c \int_{E_1}^{E_2} \sqrt{u^\alpha u_\alpha}\, d\tau$$

$$= -m_0 c^2 \int_{E_1}^{E_2} d\tau$$

$\int d\tau$ - space time distance

Geodesic - path of greatest space time distance

$$S = -m_0 c \int_{E_1}^{E_2} \sqrt{u^\alpha u^\alpha}\, d\tau$$

$$L = -m_0 c \sqrt{u^\alpha u_\alpha}$$

Non-covariant Lagrangian

$$L = -m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}}$$

$u^\alpha u_\alpha = c^2$ constraint

Soft constraint, no Lagrange multiplier required.

E_1 & E_2 are not unique functions of τ

Perform calculation assuming u^α independent and use $u^\alpha u_\alpha = c^2$ at the end.

$$S = -m_0 c \int \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}} \, d\tau$$

or $$S = -m_0 c \int \sqrt{g^{\alpha\beta} \frac{dx_\alpha}{d\tau} \frac{dx_\beta}{d\tau}} \, d\tau$$

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \left(\frac{dx_u}{d\tau} \right)} \right) - \frac{\partial L}{\partial x_u} = 0$$

$$\Rightarrow m_0 \frac{d^2 x^u}{d\tau^2} = 0$$

Charged particle in an external field

$$L = -m_0 c \sqrt{g^{\alpha\beta} \frac{dx_\alpha}{d\tau} \frac{dx_\beta}{d\tau}} - q \frac{dx_\alpha}{d\tau} A^\alpha$$

analogous to

$$L = -m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}} - q\phi + q \vec{v} \cdot \vec{A}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx_\mu}{d\tau}\right)}\right) - \partial^\mu L = 0$$

$$\Rightarrow m_0 \frac{d^2 x^\mu}{d\tau^2} + q\frac{dA^\mu}{d\tau} \quad q\frac{dx_\nu}{d\tau}\partial^\mu A^\nu = 0$$

$$\frac{dA^\mu}{d\tau} = \frac{dx_\nu}{d\tau}\partial^\nu A^\mu$$

$$m_0 \frac{d^2 x^\mu}{d\tau^2} = q\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\frac{dx_\nu}{d\tau}$$

$$P^\alpha = -\frac{\partial L}{\partial\left(\frac{dx_\alpha}{d\tau}\right)} = m u^\alpha + q A^\alpha$$

$-$ sign $\Longrightarrow$ $+$ sign

$$H = P_\alpha u^\alpha + L$$

$$H = \frac{1}{m}\left(P_\alpha - q A_\alpha\right)\left(P^\alpha - q A^\alpha\right) - c\sqrt{\left(P_\alpha - q A_\alpha\right)\left(P^\alpha - q A^\alpha\right)}$$

Hamilton's equations

$$\frac{dx^\alpha}{d\tau} = \frac{\partial H}{\partial p_\alpha} = \frac{1}{m}\left(p^\alpha - qA^\alpha\right)$$

$$\frac{dp^\alpha}{d\tau} = -\frac{\partial H}{\partial x_\alpha} = \frac{q}{m}\left(p_\beta - qA_\beta\right)\partial^\alpha A^\beta$$

We have used $\left(p_\alpha - eA_\alpha\right)\left(p^\alpha - eA^\alpha\right) = m_0^2c^2$

after differentiation

Lagrangian for continuous systems

Discrete particles

$$L = \sum_i L_i \rightarrow \int \mathcal{L}\, d^3x$$

$$S = \int L\,dt = \int \mathcal{L}\, d^4x \qquad d^4x, \text{ Lorentz scalar}$$

$\mathcal{L} \rightarrow$ Lorentz scalar $\Rightarrow$ S also scalar

$\mathcal{L}$ for the EM field is quadratic in $\vec{E}$ & $\vec{B}$, Interaction term

$$\mathcal{L}_{int} = - J_\alpha A^\alpha$$

$$\mathcal{L}_{field} \propto F_{\alpha\beta} F^{\alpha\beta} \quad \text{or} \quad \underbrace{\tilde{F}_{\alpha\beta} F^{\alpha\beta}}_{\text{pseudoscalar under inversion}}$$

$$\mathcal{L}_{field} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta}$$

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta} - J_\alpha A^\alpha$$

$$\mathcal{L} = -\frac{1}{4\mu_0} g_{\lambda\mu} g_{\nu\sigma} (\partial^\mu A^\sigma - \partial^\sigma A^\mu)(\partial^\lambda A^\nu - \partial^\nu A^\lambda) - J_\mu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{1}{4\mu_0} g_{\lambda\mu} g_{\nu\sigma} \left\{ \begin{array}{l} \delta^\mu_\beta \delta^\sigma_\alpha F^{\lambda\nu} - \delta^\sigma_\beta \delta^\mu_\alpha F^{\lambda\nu} \\ + \delta^\lambda_\beta \delta^\nu_\alpha F^{\mu\sigma} - \delta^\nu_\beta \delta^\lambda_\alpha F^{\mu\sigma} \end{array} \right.$$

$c_{\alpha\beta}$ symmetric & $F^{\alpha\beta}$ antisymmetric

$$\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)} = -\frac{F_{\beta\alpha}}{\mu_0} = \frac{F_{\alpha\beta}}{\mu_0}$$

$$\frac{\partial \mathcal{L}}{\partial A^\alpha} = -J_\alpha$$

Euler - Lagrange

$$\frac{\partial}{\partial x_\beta}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\beta A^\alpha)}\right) - \frac{\partial \mathcal{L}}{\partial A^\alpha} = 0$$

or $\partial^\beta F_{\alpha\beta} = \mu_0 J_\alpha$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2}\partial_\alpha \varepsilon^{\alpha\beta\lambda\mu} F_{\lambda\mu}$$

$$= \partial_\alpha \varepsilon^{\alpha\beta\lambda\mu}\partial_\lambda A_\mu$$

$$= \varepsilon^{\alpha\beta\lambda\mu}\partial_\alpha\partial_\lambda A_\mu$$

$$= 0$$

$\because \varepsilon^{\alpha\beta\lambda\mu}$ is antisymmetric in α and λ and $\partial_\alpha\partial_\lambda$ is symmetric

$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$ defines the fields