# PH 206: Electromagnetic theory

## Problem Set 5

Due date: Wed. Mar. 28 2012

1. The most general linear relation between the $\mathbf{D}$ and $\mathbf{E}$ vectors of an isotropic linear dielectric is

$$\mathbf{D}(\mathbf{r},t) = \int d\mathbf{r}' \int dt' \epsilon(\mathbf{r}-\mathbf{r},t-t')\mathbf{E}(\mathbf{r}',t').$$

   The magnetic permeability of the dielectric is $\mu_0$.

   (a) Write down Maxwell's equation in Fourier space, i.e. write them down in terms of $\mathbf{E}(\mathbf{k},\omega)$, $\mathbf{B}(\mathbf{k},\omega)$, $\epsilon(\mathbf{k},\omega)$, $\rho_f(\mathbf{k},\omega)$ and $\mathbf{J}_f(\mathbf{k},\omega)$, which are the Fourier transforms of the electric field, magnetic field, permittivity, free charge density and free current density.

   (b) If self-sustaining transverse electric and magnetic fields (i.e. transverse fields which are non-zero even in the absence of free charges and currents) exist in the dielectric at wavevector $\mathbf{k}$ and frequency $\omega$, what is the equation that relates $\mathbf{k}$ and $\omega$?

   (c) If self-sustaining longitudinal electric field waves exist at wavevector $\mathbf{k}$ and $\omega$, what is the relation between $\mathbf{k}$ and $\omega$? Such waves are called longitudinal plasma waves. What is the magnetic field in such a wave?

   (d) Can longitudinal magnetic field waves be set up? If so, how? If not, why?

2. Now, consider an anisotropic linear dielectric which obeys the following relation

$$D_\alpha(\mathbf{r},t) = \epsilon_{\alpha\beta}E_\alpha(\mathbf{r},t).$$

   Here $\alpha$ and $\beta$ take on the values $x$, $y$ and $z$ and the permittivity $\epsilon_{\alpha\beta}$ is thus a second rank tensor. Further, let $\epsilon_{xx} = \epsilon_{yy} = \epsilon_1$, $\epsilon_{zz} = \epsilon_2$ and $\epsilon_{xy} = \epsilon_{yx} = \epsilon_{yz} = \epsilon_{zy} = \epsilon_{zx} = \epsilon_{xz} = 0$. The permeability is $\mu_0$.

   (a) Work out the dispersion relation between the wavevector $\mathbf{k}$ and the frequency $\omega$. Show that there are two distinct dispersion relations.

   (b) Suppose a light wave propagates with a wavevector $k\hat{x}$. What are the two possible frequencies and what is the polarization associated with each?

   (c) Repeat the above exercise for a wave with wavevector $k\hat{z}$.

   (d) Suppose light of frequency $\omega$ travelling in the $\hat{x}$ direction in the dielectric has an electric field vector that oscillates in the direction $\cos\theta\hat{y} + \sin\theta\hat{z}$ at $x = 0$. What is the direction of the electric field vector's oscillations at $x = a$?

3. Consider a rectangular wave guide of side lengths $a$ and $b$. Obtain all the components of the electric and magnetic fields in the transverse magnetic $\mathrm{TM}_{m,n}$ mode. What is the cutoff frequency for each mode?

4. Calculate the Green's function for the d'Alembertian operator $\Box^2$. Remember that the Green's function is defined as

$$\Box^2 G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

   The simplest way to proceed is to use Fourier transforms. Show that one can obtain both the retarded and advanced Green's functions in this way.

5. In class we argued that the equation

$$|\mathbf{r} - \mathbf{r}_0(t')| = c(t-t'),$$

   arising in the calculation of the Liénard-Wiechert potentials of a point charge $q$ cannot have more than one solution for the retarded time $t'$ in terms of $\mathbf{r}$ and $t$ for a given trajectory $\mathbf{r}_0(t)$. However, it is possible that there are no solutions for a given trajectory at certain values of $\mathbf{r}$ and $t$. An example of such a trajectory is

$$\mathbf{r}_0(t) = \sqrt{a^2 + c^2t^2}\,\hat{z},$$

   with $-\infty < t < \infty$.

(a) Consider $\mathbf{r} = z\hat{z}$. Show that for every value of $z$, there is a value of time $\tau(z)$ such that there are no solutions for $t < \tau(z)$. Calculate $\tau(z)$. Remember, we are only interested in solutions for $t'$ which obey $t' < t$ (otherwise causality would be violated). This problem does not require complicated algebra. Draw the trajectory of the particle on a graph of $z$ vs. $ct$. From each point on the trajectory draw light rays that move *forward* in time but in either direction is space. The points of intersection of such lines with a line at fixed $z$ (which is where the potentials are going to be calculated) will give you the function $\tau(z)$.

(b) The potentials at $\mathbf{r} = z\hat{z}$ are zero for $t < \tau(z)$ since no "signal" from the point charge has reached. Calculate the potentials $\phi$ and $\mathbf{A}$ for $t > \tau(z)$.

6. A particle of mass $m$ and charge $q$ is released from infinity with speed $v$ in the direction of another point charge $Q$, which is held stationary. The interaction between the two charges is *repulsive*. Calculate the total energy radiated during the motion of the charge from infinity to the point of closest approach. Assume that the particle moves non-relativistically.