PH 206: Electromagnetic theory

Problem Set 4

Due date: Fri. Mar. 16 2012

1. Prove the continuity equation for the charge density

$$\boldsymbol{\nabla}.\mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

2. In this problem you will calculate the stress (force per unit area) on an infinitely long solenoid.

   (a) First consider an infinite sheet carrying a uniform current of density $\mathbf{K}$. Here $\mathbf{K}$ is the current per unit length normal to its direction since the geometry is two dimensional. Calculate the magnetic field due to the sheet.

   (b) Now, consider an an infinitely long solenoid carrying current I in each turn with N turns per unit length. Use the result from (a) to calculate the magnetic field at the location of a small patch of area da on the surface of the solenoid due to the rest of the solenoid. (*Hint: This calculation is analogous to that of the electric field at the location of a patch on an a spherical shell of with uniform charge density, due to the rest of the shell.*)

   (c) Having obtained the result in (b), calculate the force on the patch and show that it is proportional to its area da. In which direction is the force and what is the resultant stress?

3. A superconductor when placed in a static magnetic field develops bound currents like any other magnetic material. However these "super-currents" have a special property that their density $\mathbf{J}((\mathbf{r}) = \alpha\mathbf{A}((\mathbf{r})$, where $\mathbf{A}((\mathbf{r})$ is the magnetic vector potential and α is a constant.

   (a) What is the differential equation that the magnetic field $\mathbf{B}(\mathbf{r})$ at any point inside the superconductor obeys?

   (b) Consider a superconductor that occupies the entire region $z < 0$. Assume that there is a uniform magnetic field in the region $z > 0$ in the xy plane. Show that the magnetic field decays as one goes deeper and deeper into the superconductor (i.e. as z decreases starting from $z = 0$). What is the value of magnitude of the field at z in terms of its magnitude B_0 at $z = 0$? (*The field cannot increase going down since that would mean a very large amount of field energy and the superconductor would then rather not be a superconductor and just let the uniform field for $z > 0$ penetrate it completely.*)

   (c) On the basis of the above calculation, what will be an appropriate value for the magnetic susceptibility of a superconductor and why? You need to consider only the magnetic field deep inside the superconductor.

4. Consider two current loops 1 and 2 carrying currents I_1 and I_2 with self inductances L_{11} and L_{22} respectively and mutual inductance L_{12}. Suppose that 2 displaced by a vector $\mathbf{R}$ relative to 1. The mutual inductance $L_{12}(\mathbf{R})$ is now a function of $\mathbf{R}$.

   (a) Write down the energy $E(\mathbf{R})$ in terms of the above quantities.

   (b) The force $\mathbf{F}(\mathbf{R})$ between the loops 1 and 2 is also a function of $\mathbf{R}$. Show that $\boldsymbol{\nabla}_{\mathbf{R}}.\mathbf{F}(\mathbf{R}) = 0$ and that consequently $L_{12}(\mathbf{R})$ satisfies Laplace's equation $\boldsymbol{\nabla}^2_{\mathbf{R}} L_{12}(\mathbf{R}) = 0$.

5. This problem illustrates how the Maxwell stress tensor can be used to calculate mechanical stresses in current and charge carrying systems.

   (a) Calculate the values of all components of the Maxwell stress tensor for a spherical conductor of radius R carrying a charge Q.

   (b) Use these values to calculate the net force on a small volume element, one of whose surfaces forms a patch on the surface of the sphere and the rest of it is inside the sphere. Show that this force is proportional to the area of the patch and radially outwards and the resultant stress is the same calculated from simple electrostatics.

(c) Now, calculate all components of the Maxwell stress tensor for an infinitely long solenoid with N turns per unit length each carrying a current I.

(d) Use your calculation from (c) to calculate the force on a patch of area da on the surface of the solenoid. Show that the force is the same you obtained in 2(c) and thus also the stress.

6. In class, we argued that the angular momentum carried by electromagnetic fields can be converted to "mechanical" angular momentum. To see this more clearly, consider a static distribution of charges of density $\rho(\mathbf{r})$ placed in a static magnetic field $\mathbf{B}(\mathbf{r})$.

(a) How is the electric field $\mathbf{E}(\mathbf{r})$ generated by the charges related to $\rho(\mathbf{r})$? What is $\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r})$? What is the total angular momentum?

(b) Now, assume that the the magnetic field is changed by an amount $\Delta\mathbf{B}(\mathbf{r})$ in time Δt. What relation does the electric field $\mathbf{E}'(\mathbf{r})$ produced by this change in the magnetic field obey? What is $\boldsymbol{\nabla}.\mathbf{E}'(\mathbf{r})$?

(c) $\mathbf{E}'(\mathbf{r})$ will exert a torque on the charge distribution. What is the "mechanical" angular momentum ΔL_M imparted in time Δt? What is the change in the angular momentum of the fields ΔL_F? Express your answers as suitable integrals over $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}'(\mathbf{r})$.

(d) Show that $\Delta L_M = -\Delta L_F$. To do this, use the expressions for the divergence and curls of $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}'(\mathbf{r})$ and vector identities. You will also have to use the fact that $\mathbf{E}(\mathbf{r})$ and $\mathbf{E}'(\mathbf{r})$ fall off as $1/r^2$ or faster as $r \to \infty$ which is certainly true if the static charge and source of magnetic field are localized in space.