PH 206: Electromagnetic theory

Problem Set 2

Due date: Tue. Feb 14 2012

1. A cube of side a with no charges inside has two opposite faces at potentials V_1 and V_2 and all other faces at zero potential. Calculate the potential everywhere inside the cube.

2. The potential on a circle of radius R is given by $V(\theta)$, where θ is the angle with respect to the x axis which passes through the centre of the circle.

   (a) What does the general solution to Laplace's equation look like inside and outside the circle?

   (b) Show that there is a conformal transformation that maps the interior of the circle to its exterior and hence the solution for the exterior can be obtained by knowing the solution in the interior. Verify that the solution that you obtain for the exterior this way matches the one you obtained in (a). (*Hint: It might help to think of the interior of the circle as the annular region between the circle and another circle or radius ϵ sharing the same centre and then taking ϵ to zero.*)

3. Poisson's equation is

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

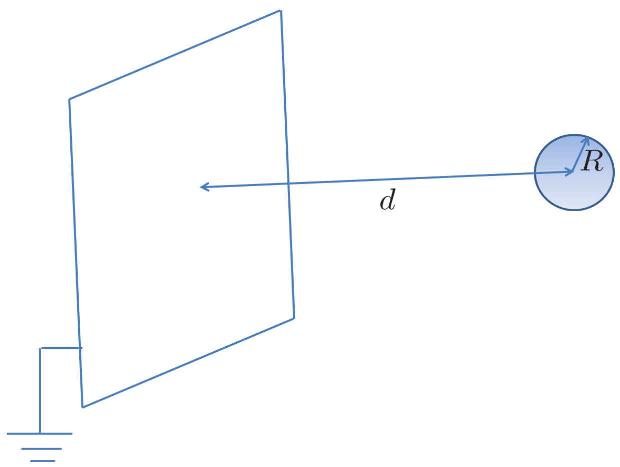
   In class it was argued that there exists a Green's function for this equation satisfying

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}').$$

   (a) Consider the case of three dimensions with Poisson's equation having the boundary condition $\phi \to 0$ as $r \to \infty$. Show that $G(\mathbf{r}, \mathbf{r}') = 1/|\mathbf{r} - \mathbf{r}'|$. (*Hint: Use Fourier transforms.*)

   (b) Now, consider a one dimensional situation with the boundary condition $\phi(x = L) = \phi(x = -L) = 0$. Calculate the Green's function $G(x, x')$. What happens as $L \to \infty$?

4. A conducting sphere of radius R is kept a distance d away from an infinite grounded plane (i.e. the centre of the sphere is a distance d away from the plane) as shown in the figure. In this problem you will calculate the capacitance of such a sphere using the method of images.



   (a) Assume that the surface of the sphere is at a potential V. What is the potential in the interior of the sphere? What is the potential everywhere on the side of the plane that does not contain the sphere, i.e. to the left of the plane in the figure?

(b) To calculate the potential in the region exterior to the sphere and on the side of the plane where the sphere is (the right of the plane in the figure) first put a charge q_0 at the centre of the sphere. This makes the sphere an equipotential but not the plane. To make the plane a zero potential surface place the image charge q_1 of q_0 on the other side of the plane. Now, the sphere is no longer an equipotential so we need to place the image charge q_2 of q_1 inside the sphere at the appropriate place to make it an equipotential. But, as a result the plane is again not at zero potential so we need the image charge of q_2 and so on. Thus, the method of images here requires an infinite number of image charges. Show that the potential V of the surface of the sphere and Q, the total charge on it are both proportional to q_0. Argue that $V = \frac{q_0}{4\pi\epsilon_0 R}$ and $Q = f(d/R)q_0$, where $f(x)$ is obtained by summing the appropriate infinite series in x. What is the value of $f(\infty)$? The capacitance $C = 4\pi\epsilon_0 R f(d/R)$.

(c) Sum the series for $f(x)$ numerically. This involves summing only a finite number of terms in the infinite series corresponding to a finite number of image charges. Obtain $f(x)$ for x between 0 and 5 and plot it as functions of x. Sum as many terms as required to get convergence to within 0.001% for all x between 0 and 5. Check that $f(x)$ approaches the value of $f(\infty)$ that you calculated in (b) numerically by performing the claculation for large x.

5. Your mischievous (and possibly sadistic) friends decide to play a prank on you by rubbing a woollen cloth vigorously on a door knob they know you will touch so you get a static shock. Assume the door knob is spherical so that the geometry of the previous problem applies. The grounded infinite conducting plane is the floor.

(a) Assuming typical dimensions (the door knob is a few centimetres in diameter and is about a metre or so above the floor), argue that the voltage of the door knob can be of the order of 10,000 volts or even larger. (*Hint: Roughly how much charge would rubbing the door knob with wool produce and why? Remember that the door knob is made of a metal and use appropriate values of the required physical parameters making reasonable approximations.*)

(b) Why does touching a door knob at such a high voltage only produce a mildly irritating shock whereas if you were to touch a live wire at 220 volts, the electric shock would be much more severe?

6. Consider a system of "free" charges $\{q_i\}$ and "bound" charges $\{Q_i\}$. Both types of charges feel the Coulomb force due to all other charges (free and bound). However, the bound charges feel additional "mechanical" forces due to the other bound charges. Let the mechanical force between a pair of bound charges i and j be obtained from a potential $U_{ij}(\mathbf{R}_i - \mathbf{R}_j)$, where $\mathbf{R}_i$ and $\mathbf{R}_j$ are the position vectors of the two bound charges. You will show that the total energy of this system can be determined by calculating the work done to assemble *only the free charges*. As the free charges are brought from infinity, the bound charges respond by arranging themselves so that the net Coulomb force on them is balanced by the next mechanical force. Thus the positions of the bound charges $\{\mathbf{R}_i\}$ are functions of the positions of the free charges $\{\mathbf{r}_i\}$.

(a) Suppose the free charges $\{q_i\}$ are at $\{\mathbf{r}_i\}$ and the bound charges $\{Q_i\}$ are at $\{\mathbf{R}_i\}$. What is the total electrostatic energy of the system? What is the total mechanical energy?

(b) What is the total work done to assemble all the free charges?

(c) Show that the work done obtained in (b) is the sum of the total electrostatic energy and mechanical energy obtained in part (a).

(*Hint: Imagine that $N-1$ free charges have already been brought to their final positions $\{\mathbf{r}_i\}$. Calculate the work done W_N to bring the N^{th} free charge to its final position $\mathbf{r_N}$. $W_N = -\int_{\infty}^{\mathbf{r_N}} \mathbf{F_N}(\mathbf{r}').\mathbf{dr}'$, where $\mathbf{F_N}(\mathbf{r}')$ is the total force on the N^{th} charge when it is at position $\mathbf{r}'$. $\mathbf{F_N}(\mathbf{r}')$ is purely Coulombic in nature and has two pieces, one from the other $N-1$ free charges and another from all the bound charges. Use the fact that the positions of the bound charges depend on the instantaneous positions of the free charges and the net force on each bound charge is zero throughout the process of bringing the N^{th} free charge from infinity.*)