PH 206: Electromagnetic theory

Problem Set 0

Due date: Fri. Jan 13 2012

1. A vector field $\mathbf{F}(\mathbf{r})$ obeys $\boldsymbol{\nabla}\times\mathbf{F} = 0$ in a volume Ω. Show that this implies there exists a scalar field $\phi(\mathbf{r})$ such that $\mathbf{F} = -\boldsymbol{\nabla}\phi$ in Ω.

2. Consider the two dimensional space $\mathbb{S} = \mathbb{R}^2 - (0,0)$, which is just the entire xy plane with the origin removed. Let us define $\phi(x,y) = \tan^{-1}(y/x)$ on $\mathbb{S}$.

   (a) Show that it is not possible for $\phi(x,y)$ to be single valued and $\boldsymbol{\nabla}\phi$ to exist everywhere on $\mathbb{S}$.

   (b) Let us demand that $\boldsymbol{\nabla}\phi$ exists everywhere on $\mathbb{S}$. What is an appropriate choice of the multi-valued function $\phi(x,y)$ for this to happen? What $\boldsymbol{\nabla}\phi$ does this give? (*Hint: It might help to think in terms of polar co-ordinates.*)

   (c) Let $\mathbf{F} = -\boldsymbol{\nabla}\phi$ for the ϕ obtained in the previous part. What is $\boldsymbol{\nabla}\times\mathbf{F}$?

   (d) Calculate the line integral

   $$\oint \mathbf{F}.d\mathbf{l},$$

   along any circle with centre at $(0,0)$. Does the result you get seem to contradict Stokes' theorem,

   $$\oint \mathbf{F}.d\mathbf{l} = \iint \boldsymbol{\nabla}\times\mathbf{F}.d\mathbf{S},$$

   where the surface integral is over a region bounded by the circle? If so, how will you "fix" Stokes' theorem?

   (e) Now let $\mathbb{S} = \mathbb{R}^2$, i.e. $\mathbb{S}$ is the xy plane *including* the origin and $\phi(x,y)$ is the same as in part (b). Is there a contradiction of Stokes' theorem? If so, how will you fix it? What is $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi$?

3. A particle of mass m and charge q moves in a magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ both of which are *spatially uniform* and *time independent*. At time $t = 0$, the particle has displacement $\mathbf{r}_0$ and velocity $\mathbf{v}_0$.

   (a) What is the displacement $\mathbf{r}(t)$ at a later time t?

   (b) What does the trajectory of the particle look like if $\mathbf{E} \parallel \mathbf{B}$?

   (c) What does the trajectory look like if $\mathbf{v}_0$, $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to one another?

4. A circular wire of radius R carries a current I. $\mathbf{B}(\mathbf{r})$ is the magnetic field it produces.

   (a) What is the value of the line integral

   $$\int \mathbf{B}.d\mathbf{l},$$

   which is calculated along the axis of the wire, passing through its centre, perpendicular to its plane and extending from $-\infty$ to $+\infty$?

   (b) What is the value of the line integral if it is calculated along a line inclined at $45°$ to the axis in part (a) again extending from $-\infty$ to $+\infty$?

5. Let

$$\mathbf{E}_0(\mathbf{r},t) = A\left\{\cos\left[k_z\left(z-ct\right)\right]\hat{x} + \cos\left[k_x\left(x-ct\right)\right]\hat{y} + \cos\left[k_y\left(y-ct\right)\right]\hat{z}\right\},$$

$$\mathbf{B}_0(\mathbf{r},t) = \frac{A}{c}\left\{\cos\left[k_y\left(y-ct\right)\right]\hat{x} + \cos\left[k_z\left(z-ct\right)\right]\hat{y} + \cos\left[k_x\left(x-ct\right)\right]\hat{z}\right\},$$

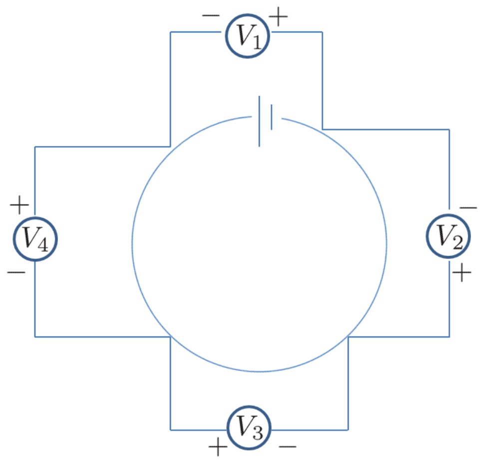
$$\mathbf{U}(\mathbf{r}) = \cos(k_x x)\,\hat{x} + \cos(k_y y)\,\hat{y} + \cos(k_z z)\,\hat{z},$$

$$\mathbf{V}(\mathbf{r}) = \cos(k_y y)\,\hat{x} + \cos(k_z z)\,\hat{y} + \cos(k_x x)\,\hat{z},$$

where A, k_x, k_y and k_z are real numbers and $c = 1/\sqrt{\mu_0 \epsilon_0}$. Which among the following combinations of electric and magnetic fields are not valid and why? Note that sources of charge and current are allowed to exist.

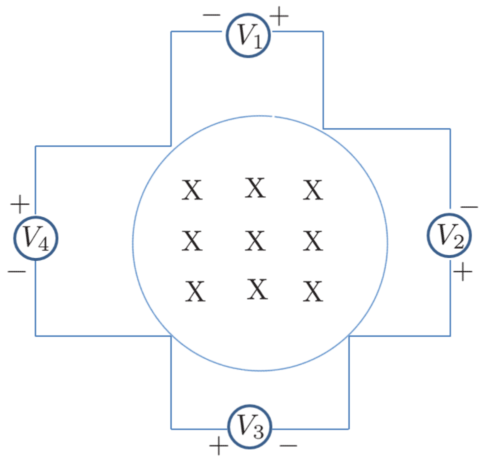
(a) $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t)$
(b) $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t) + \frac{A}{c}\mathbf{U}(\mathbf{r})$
(c) $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r},t) + A\mathbf{V}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t)$
(d) $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r},t) + A\mathbf{U}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r},t) + \frac{A}{c}\mathbf{V}(\mathbf{r})$

6. A wire of resistance per unit length ρ is bent into a circle of radius R and connected to a battery of voltage V as shown in the figure below. The dimensions of the battery may be considered to be very small compared to the length of the wire. Four voltmeters are connected around the circuit as shown. $+$ and $-$ indicate the polarities of their leads.



(a) What is the current flowing through the wire?
(b) What is $V_1 + V_2 + V_3 + V_4$?

Now consider a different situation in which the same circular wire is placed in a uniform time dependent magnetic field $B(t) = \alpha t$ perpendicular to its plane as shown in the figure below (The X's indicate the magnetic field).



(c) What is the value of α so that the current in the wire is the same as in part (a)?
(d) What is $V_1 + V_2 + V_3 + V_4$?