- Exact Integration.
- Simple numerical methods.
- Advanced numerical methods.

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- Computer Algebra Systems (CAS).

Computer Algebra Systems

- Mathematica
- Matlab
- Sympy
- Magma
- Sagemath
- etc.

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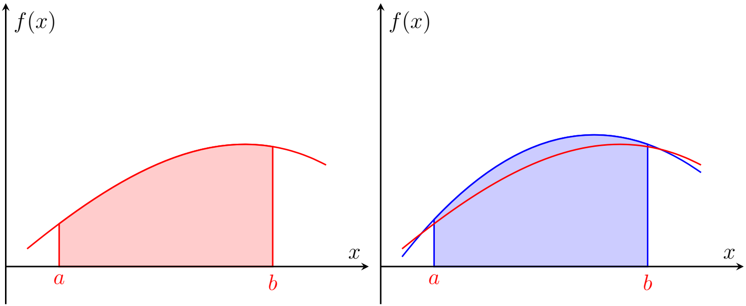
- The data that we have collected and need to integrate is discrete data – ie is only known at some points (we may or maynot know the functional form).

Numerical integration can be based on fitting approximating functions (polynomials) to discrete data and integrating approximating functions.

$$I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$



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- The parameters under the users control are:
  - The degree of the approximating polynomial.
  - The total number of discrete points used in the calculation of the integral.
  - The location of the points where the function is discretized.

Direct fit polynomials

- This procedure is based on the idea that one can fit the data by a direct fit polynomial and integrate that polynomial:

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- This procedure can be applied to data ($f(x)$) which is available for x that are equally spaced or unequally spaced.

- If $f(x)$ is a continuous function defined for $a \leq x \leq b$ and one divides the interval $[a, b]$ into n subintervals of equal width, $\Delta x = \frac{b-a}{n}$ then the definite integral

$$I = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $f(x_i^*)$ is the value of the function at an arbitrary point, x_i^*, in the interval x_i and $x_i + \Delta x$.

Riemann Integral

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- The Riemann integral can be interpreted as the area under the curve $y = f(x)$ from a to b.

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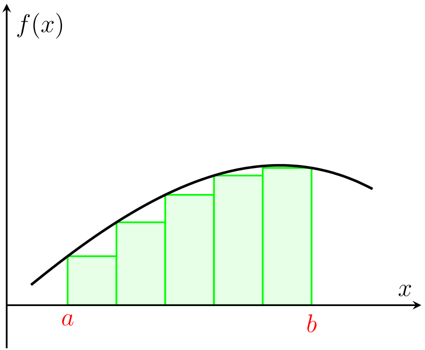

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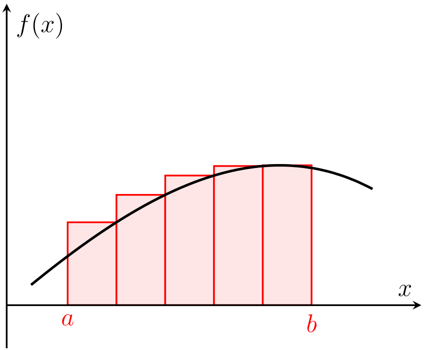

Mid point Riemann sum

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$$I = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x \quad \Delta x = \frac{b-a}{n}$$

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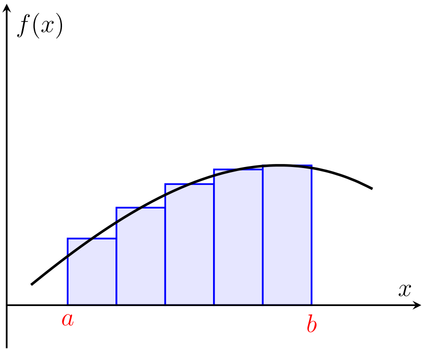
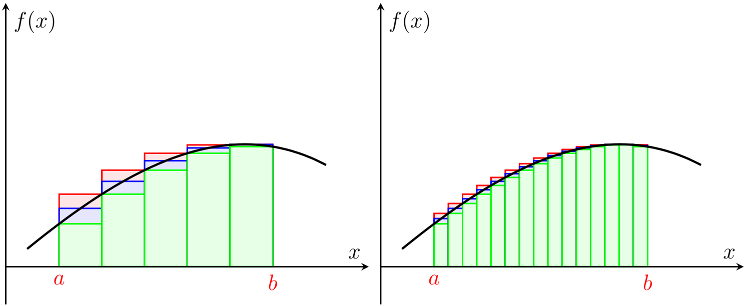


Comparison of left, right and mid point



In principle, all three of the methods will converge to the same result – albiet very slowly!

Better method: Trapezoidal approximation

The area of the trapezoid that lies above the i^{th} subinterval:

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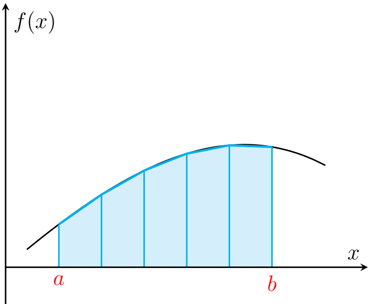
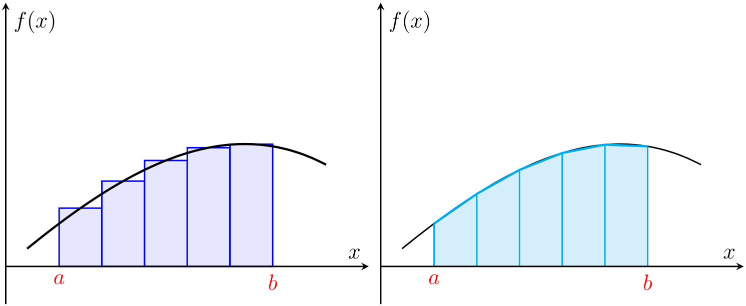


Comparison of mid point and trapezoidal



First order interpolation

First order interpolation for the i^{th} subinterval:

$$\begin{aligned} f(x) =& f(x_{i-1}) + f'(x_{i-1})x + higher\ order\ terms \\ =& f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{\Delta x} x + higher\ order\ terms. \end{aligned}$$

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Trapezoidal approximation is the application of first order interpolation for each subinterval.

Integration with second order interpolation

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Number of n intervals should be even – if not then the last interval should be treated in some other way!