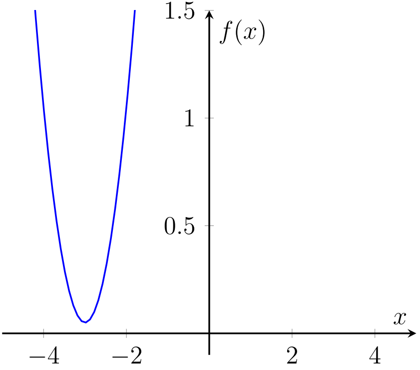
- Complications.
- Roots of polynomials.
- Non-linear systems of equations with multiple variables.
- Summary.

Complications

- There are no roots at all!

The hardest thing of all is to find a black cat in a dark room, especially if there is no cat.

Confucius.



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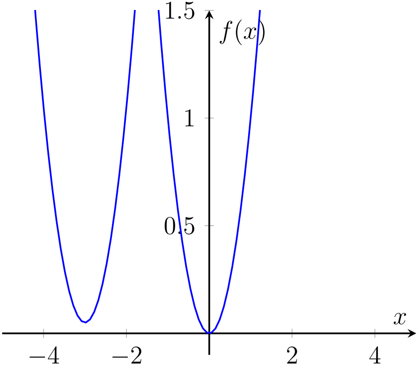

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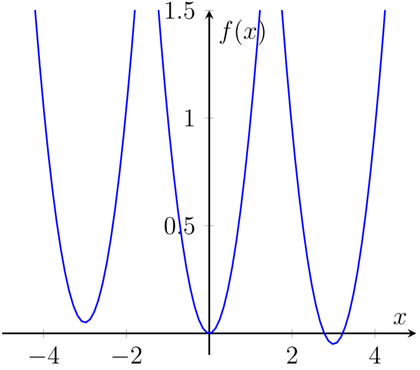
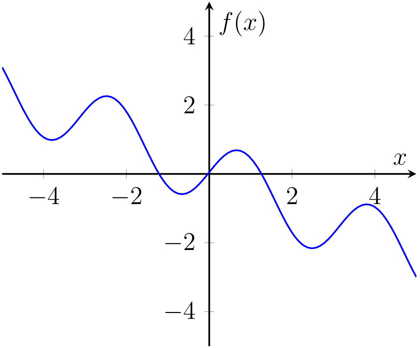
- There is one root but the function does not change sign.
- There are two or more roots in the interaval $[a, b]$.



More complications

There are many roots.



What will happen if you use bracketing methods for this?

Multiple roots: Brute force method

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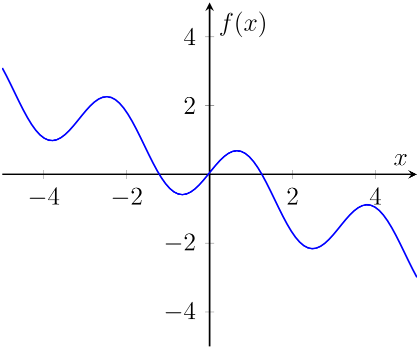

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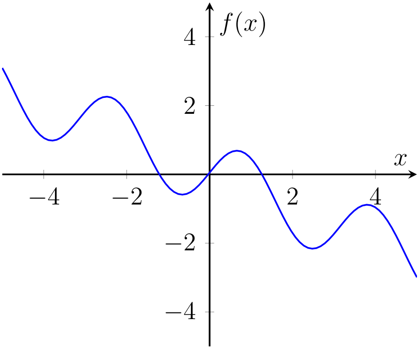
- One splits the original interval $[a, b]$ into smaller intervals with some step size (say h) and then applies the previously mentioned methods in each of the sub-intervals.



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- A graphical analysis is very helpful in deciding the step size h.
- *It may be good to evaluate roots with h and then with $h/10$ to confirm that the number of roots remains unchanged.*

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- If the coefficients are all real, then the complex roots always occur in conjugate pairs.
- The roots may be simple (ie. single) or repeated (ie multiple).

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- They can however be used to find repeated roots with odd multiplicity.
- Open domain methods are often more efficient than closed domain methods.

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- For complex roots – one just needs to use complex arithemetic – the algorithms remain the same.
- There are various modifications of the Newton's method which are often used to find roots. For eg. Bairstow's method.

Non linear system of equations: two variables

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$$f(x, y) = 0$$
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- Open domain methods (especially Newton's method) are easily extended to solve such systems.
- One always needs a good guess!!!

Extension of Newton's method to two variables

We are given two equations:

$$f(x, y) = 0$$
$$g(x, y) = 0$$

We have to find the solution (x^*, y^*) such that:

$$f(x^*, y^*) = 0$$
$$g(x^*, y^*) = 0$$

Taylor expanding about (x^*, y^*):

$$f(x, y) = f(x^*, y^*) + (x - x^*)f'_x + (y - y^*)f'_y + \dots$$
$$g(x, y) = g(x^*, y^*) + (x - x^*)g'_x + (y - y^*)g'_y + \dots$$

Extension of Newton's method to two variables

Keeping only first-order terms and given $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$, one has a system of linear equations for x^* and y^* such that:

$$x^* = x + \frac{f'_y g(x,y) - g'_y f(x,y)}{f'_x g'_y - f'_y g'_x}$$
$$y^* = y + \frac{g'_x f(x,y) - f'_x g(x,y)}{f'_x g'_y - f'_y g'_x}$$

This method can be easily generalized to solving n non linear equations.

Non linear systems of equations: Example

$$f(x,y) = y^2(1-x) - x^3 = 0$$
$$g(x,y) = y^2 + x^2 - 1 = 0$$

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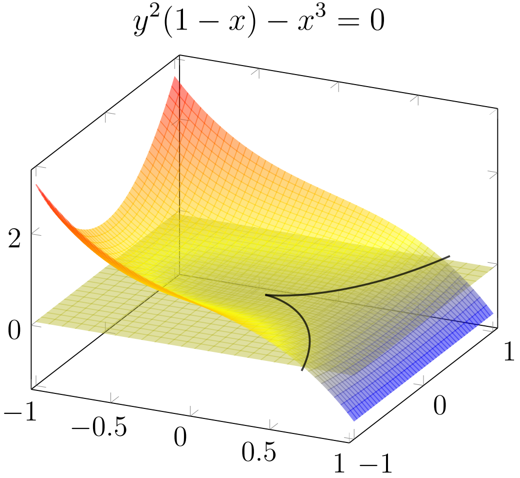

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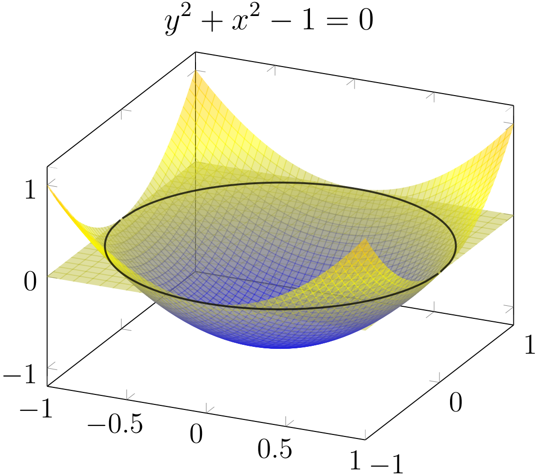

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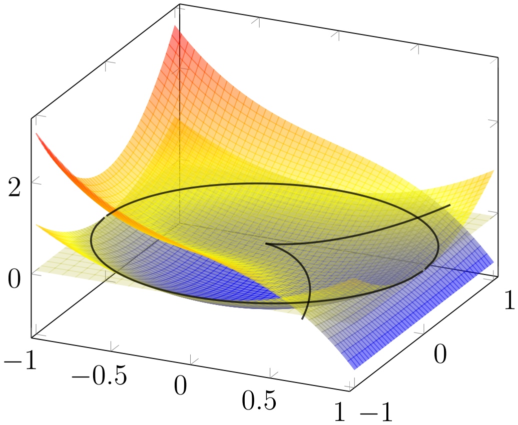

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Newton's method: example

i	x	y	f(x,y)	g(x,y)	dx	dy
1	1.000000	1.000000	-1.000000	1.000000	-0.250000	-0.250000
2	0.750000	0.750000	-0.281250	0.125000	-0.119048	0.035714
3	0.630952	0.785714	-0.023352	0.015448	-0.012757	0.000414
4	0.618195	0.786128	-0.000298	0.000163	-0.000161	0.000023
5	0.618034	0.786151	-0.000000	0.000000		

Root is: 0.6180340120481262, 0.7861513762373674

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- Trade off between your time and computer's time...
- Choosing the right method is difficult – something that works in one equation can miserably fail in another.
- If possible, visualize the data – this helps a lot.