Roots of Equations

- Real roots of single variable function.
- Closed domain methods (bracketing).

Examples of nonlinear equations

- Equations of a single variable:

$$x^2 - 6x + 9 = 0$$
$$x^2 - \cos(x) = 0$$
$$\exp(x)\ln(x^2) - x\cos(x) = 0$$

- Equations of two variables:

$$y(x^3 - 1) = x^4$$
$$x^2 + y^2 = 1$$

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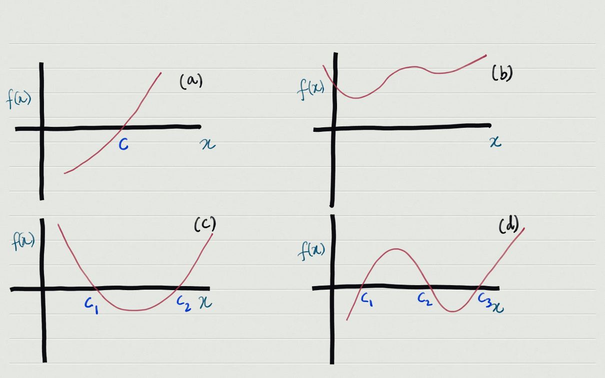
Behaviour of non-linear functions



(a) A single root.
(b) No real roots exist (complex roots might).
(c) Two simple roots.
(d) Three simple roots.

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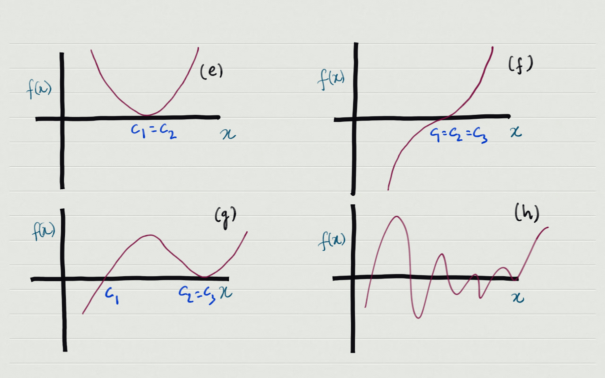

(e) Two multiple roots.
(f) Three multiple roots.
(g) One simple root and two multiple roots.
(h) Multiple roots.

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- Starting from the guess we iterate – the better the guess, the better our chances of converging to the solution and fewer the number of iterations required to converge to the solution.
- So, it is crucial to have a good guess...

Bounding the solution

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- If possible, the root should be bracketed between two points at which the value of the non-linear function changes sign!

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- Previous solution in a sequence of solutions.

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- *There are numerous pitfalls in finding the roots of non linear equations*.
- An important question is when to stop the iteration –

$$\text{Absolute error:} |f_{i+1} - f_i|$$

$$\text{Relative error:} |\frac{f_{i+1} - f_i}{f_{i+1}}|$$

Types of Methods

There are two types of methods for finding roots of non linear equations:

- Closed domain (bracketing) methods.
- Open domain (non-bracketing) methods.

Closed domain (bracketing) method

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- Most common closed domain methods:
  - Bisection method – interval halving.
  - False position method.
- In general, bracketing methods are quite robust – ie they are guaranteed to give a solution as the solution is bracketted in the interval.

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- $f(x)$ a continuous function on $[a, b]$.
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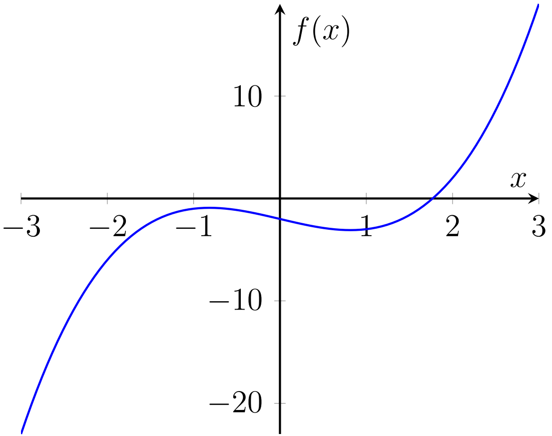
Consider an example: $f(x) = x^3 - 2x - 2$



Bisection Method: Algorithm

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$$f(a)f(c) \begin{cases} < 0 \text{ there is a root in } [a, c] \implies a = a, b = c \\ > 0 \text{ there is a root in } [b, c] \implies a = c, b = b \\ = 0 \text{ root is at } c \end{cases}$$

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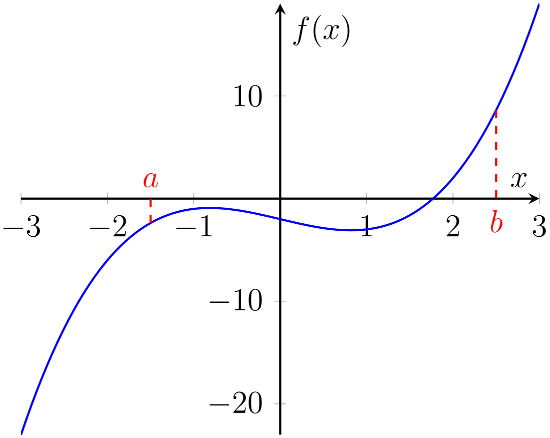
- Interval halving is an iterative procedure.
- Continue iterations till $|b - a| < \text{tol}$ or $|f(c)| < \text{tol}$ or both

Bisection Method: Example



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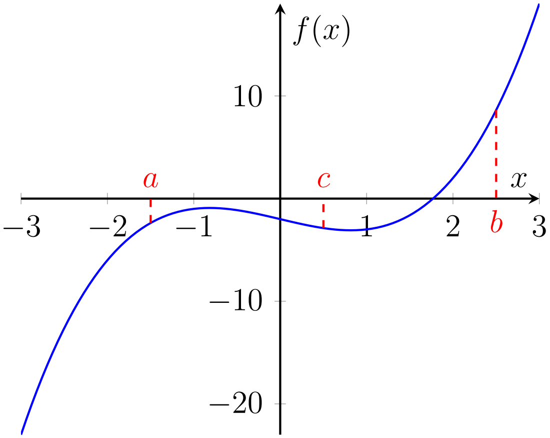

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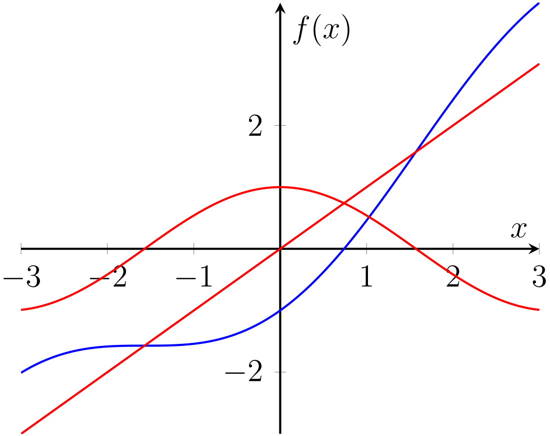
- However, the convergence of the method can be quite slow.
- Notice that this method does not use any information about the function's behaviour!

Bisection method: Example

Consider an example: $f(x) = x - \cos(x)$



Bisection method: example

i	a	f(a)	b	f(b)	c	f(c)
1	0.000000	-1.000000	4.000000	4.653644	2.000000	2.416147
2	0.000000	-1.000000	2.000000	2.416147	1.000000	0.459698
3	0.000000	-1.000000	1.000000	0.459698	0.500000	-0.377583
4	0.500000	-0.377583	1.000000	0.459698	0.750000	0.018311
5	0.500000	-0.377583	0.750000	0.018311	0.625000	-0.185963
6	0.625000	-0.185963	0.750000	0.018311	0.687500	-0.085335
7	0.687500	-0.085335	0.750000	0.018311	0.718750	-0.033879
8	0.718750	-0.033879	0.750000	0.018311	0.734375	-0.007875
9	0.734375	-0.007875	0.750000	0.018311	0.742188	0.005196
10	0.734375	-0.007875	0.742188	0.005196	0.738281	-0.001345
11	0.738281	-0.001345	0.742188	0.005196	0.740234	0.001924
12	0.738281	-0.001345	0.740234	0.001924	0.739258	0.000289
13	0.738281	-0.001345	0.739258	0.000289	0.738770	-0.000528
14	0.738770	-0.000528	0.739258	0.000289	0.739014	-0.000120
15	0.739014	-0.000120	0.739258	0.000289	0.739136	0.000085
16	0.739014	-0.000120	0.739136	0.000085	0.739075	-0.000017
17	0.739075	-0.000017	0.739136	0.000085	0.739105	0.000034
18	0.739075	-0.000017	0.739105	0.000034	0.739090	0.000008
19	0.739075	-0.000017	0.739090	0.000008	0.739082	-0.000005
20	0.739082	-0.000005	0.739090	0.000008	0.739086	0.000002
21	0.739082	-0.000005	0.739086	0.000002	0.739084	-0.000001
22	0.739084	-0.000001	0.739086	0.000002	0.739085	0.000000

```
Root is: 0.739085197449
```

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- The root of the linear function, $g(x)$, $x = c$ is not the root of the non linear function $f(x)$. It is a *false position* and hence the name.
- Unlike bisection, this method uses some information about the function $f(x)$.

False position method: Algorithm

The slope of linear function, $g'(x)$, is given by:

$$g'(x) = \frac{f(b) - f(a)}{b - a}$$

Assuming $f(c) = 0$, one can also write $g'(x)$ as:

$$g'(x) = \frac{f(b) - f(c)}{b - c} \implies c = b - \frac{f(b)}{g'(x)}$$

Combining with the first equation:

$$c = b - f(b) * \frac{b - a}{f(b) - f(a)}$$

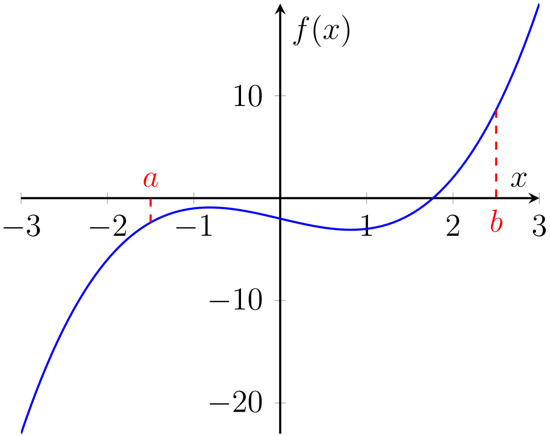
$$c = \frac{a * f(b) - b * f(a)}{f(b) - f(a)}$$

Then just like the bisection method:

$$\text{if } f(a) * f(c) < 0 \; a = a, b = c$$

$$\text{if } f(a) * f(c) > 0 \; a = c, b = b$$

False position method: Example



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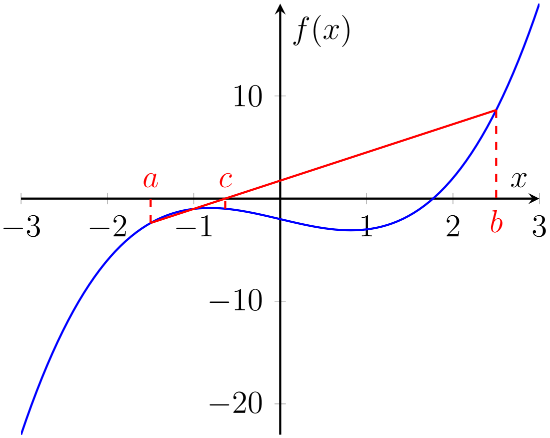

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3	0.707508	-0.052475	0.744221	0.008605	0.739049	-0.000061
4	0.739049	-0.000061	0.744221	0.008605	0.739085	-0.000000

Root is: 0.739085092149

False position used 4 iterations compared to 22 in bisection!
Generally false position converges much faster than bisection.