Ordinary Differential Equations

- Boundary value problems.
- Shooting method.
- Equilibrium boundary value method/ Finite difference method.

Boundary value problems

- Boundary-value problems – involve differential equations with specified boundary conditions: example: one-dimension second order ODE (where p and q are some constants)

$$\frac{d^y}{dx^2} + p\frac{dy}{dx} + q = f(x)$$

with $y(x_1) = y_1$ and $y(x_2) = y_2$.

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  with $y(x_1) = y_1$ and $y(x_2) = y_2$.
- The boundary-value problem is more difficult to solve than the similar initial-value problem with the same differential equation.

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- The function $y(x)$ is specified – Dirichlet boundary conditions
- The derivatives $y'(x)$ is specified – Neumann boundary conditions
- A combination of $y(x)$ and $y'(x)$ is specified – mixed boundary conditions

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- Only boundary condition on one side is used as one of the initial conditions. The additional initial condition is assumed.
- Then an iterative approach is used to vary the assumed initial condition till the boundary condition on the other side is satisfied.

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- Quite often the fourth-order Runge-Kutta is combined with the secant method.

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- Using secant method for root finding:

$$z_{k+1} = z_k - \frac{c_k - y_2}{c_k - c_{k-1}}(z_k - z_{k-1})$$

The equilibrium boundary-value problem

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- Approximating the exact derivatives in the boundary-value ODE by algebraic finite difference approximations.
- Substituting the FDAs into the ODE to obtain an algebraic finite difference equation.
- Solving the resulting system of algebraic FDEs (for linear ODEs – a system of linear equations)

The equilibrium boundary-value problem

- Consider a second order, linear, variable coefficient, boundary value problem with Drichlet boundary conditions:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = F(x)$$

with $y(x_1) = y_1$ and $y(x_N) = y_N$

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- Discretizing the domain of x into N points $x_1, x_2, \ldots, x_N$
- The second order centered difference approximation for y' and y'':

$$y'(x_i) = y'_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$
$$y''(x_i) = y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

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- Substituting this into the differential equation and keeping terms upto order $\mathcal{O}(\Delta x^2)$:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + P_i\left(\frac{y_{i+1} - y_{i-1}}{2\Delta x} + \right) + Q_i y_i = F_i$$

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$$\left(1 - \frac{\Delta x}{2} P_i\right) y_{i-1} + (-2 + \Delta x^2 Q_i) y_i + \left(1 + \frac{\Delta x}{2} P_i\right) y_{i+1} = \Delta x^2 F_i$$

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- Number of equations $= N - 2$ – can solve this easily with a linear equation solver.

Comment on higher order BVP

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- Example 4th oder problem:

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  and 4 boundary conditions.
- For first/second derivative: $i-1, i, i+1$
  For third/fourth derivative: $i-2, i-1, i, i+1, i+2$
- Leads to a pentadiagonal system.

Neumann boundary condition

- Consider a second order, linear, variable coefficient, boundary value problem with Neumann boundary conditions:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = F(x)$$

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- The equilibrium method needs some modification:

$$\left(1-\frac{\Delta x}{2}P_i\right)y_{i-1}+(-2+\Delta x^2 Q_i)y_i+\left(1+\frac{\Delta x}{2}P_i\right)y_{i+1} = \Delta x^2 F_i$$

remains the same at all the interior points.

Neumann boundary condition

- We also apply this equation to the boundary point:

$$\left(1 - \frac{\Delta x}{2} P_N\right) y_{N-1} + (-2 + \Delta x^2 Q_N) y_N + \left(1 + \frac{\Delta x}{2} P_N\right) y_{N+1} = \Delta x^2 F_N$$

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- The point y_{N+1} is outside the domain but:

$$\begin{aligned} y'_N &= \frac{y_{N+1} - y_{N-1}}{2\Delta x} \\ y_{N+1} &= y_{N-1} + 2\Delta x y'_N \end{aligned}$$

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- Substituting this in the equation for N point:

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- Again a tridiagonal system of equations.

Boundary condition at infinity

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- Replace ∞ with a large value of x ($x = X$)
- Asymptotic solution at large values of x

Higher order methods

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- For shooting method – it is quite clear how one can use a higher order method. There is hardly any limitation.
- For finite differencing, one has to go to higher order finite differencing to get a better y' and y''. Higher order finite differencing formulae exist, but the coding becomes a bit messier.
- Problem with the method near the boundaries.. central differencing cannot be applied near the boundaries. So, some forward/backward differencing is used there.

Non linear boundary value problems

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- With the shooting method, the solution of nonlinear boundary value problems is quite straighforward.
- With the finite differencing, this is much more complicated as the corresponding Finite differencing Equation is non-linear. This leads to a non-linear system of FDE's. Have to use Newton's iteration method to solve these.

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- By either approach, the solution is quite difficult.