Ordinary Differential Equations

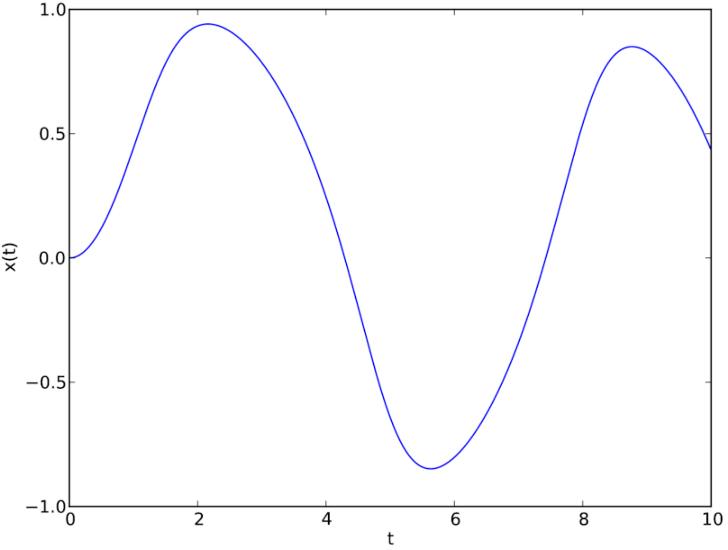
- Euler's method.
- Runge Kutta methods.
- Simultaneous differential equations.

Euler's method

Let us use the Euler's method to solve the differential equation:

$$\frac{dx}{dt} = -x^3 + \sin t$$

Euler's method – output



This is a reasonable representation of the actual solution.

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- Technically, Euler's method is the first-order Runge-Kutta method.
- Consider the next method in the series – the second-order Runge-Kutta method.

Runge-Kutta methods

$$\frac{dx}{dt} = f(x, t)$$

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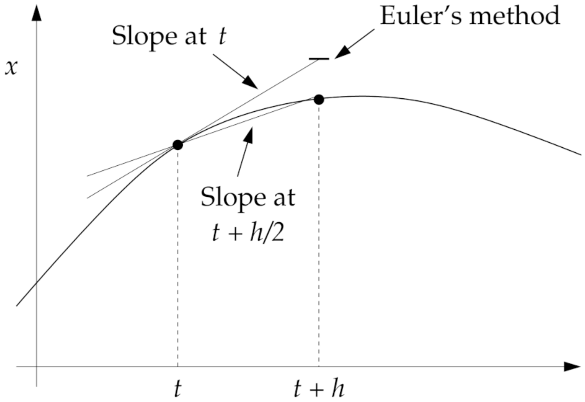
- Euler's method can be represented as:



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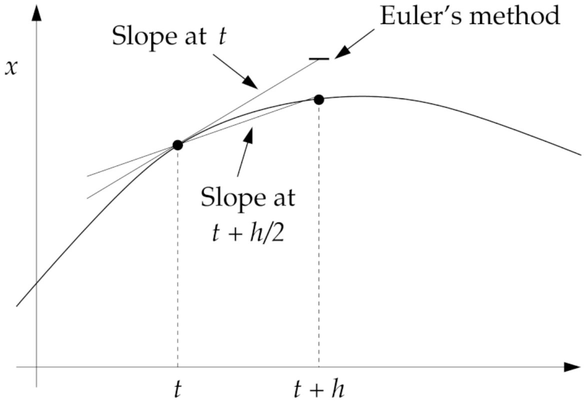

- If we use the slope at $t + \frac{1}{2}h$ to extrapolate, we do better!

Runge-Kutta methods

- Performing the Taylor expansion arount $t + \frac{1}{2}h$:

$$x(t+h) = x(t+\tfrac{1}{2}h)+\tfrac{1}{2}h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h}+\tfrac{1}{8}h^2\left(\frac{d^2x}{dt^2}\right)_{t+\frac{1}{2}h}+\mathcal{O}(h^3)$$

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$$\begin{aligned} x(t+h) &= x(t) + h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h} + \mathcal{O}(h^3) \\ &= x(t) + hf(x(t+\tfrac{1}{2}h), t+\tfrac{1}{2}h) + \mathcal{O}(h^3) \end{aligned}$$

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Error is now $\mathcal{O}(h^3)$! Better than Euler ($\mathcal{O}(h^2)$).

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- Then the whole algorithm becomes:

$$\begin{aligned} k_1 &= hf(x, t) \\ k_2 &= hf(x + \tfrac{1}{2}k_1, t + \tfrac{1}{2}h) \\ x(t + h) &= x(t) + k_2 \end{aligned}$$

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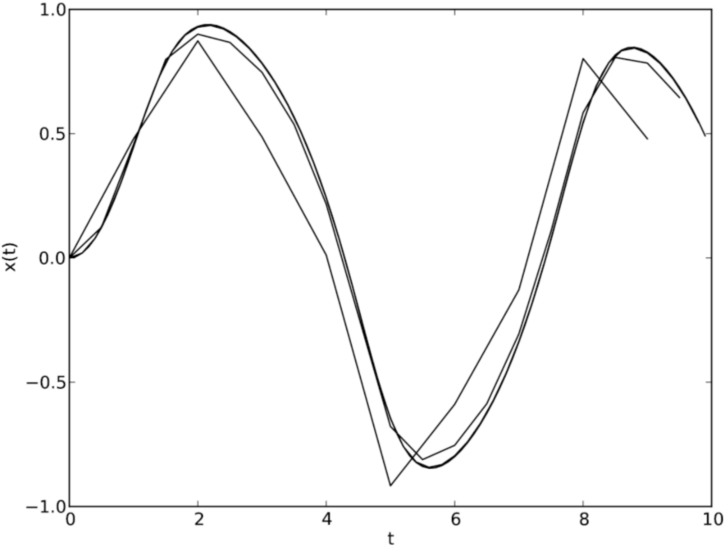
- Error in each step is $\mathcal{O}(h^3)$ and global error is $\mathcal{O}(h^2)$.

Second order Runge-Kutta method

Let us use the second-order Runge-Kutta method to solve the differential equation:

$$\frac{dx}{dt} = -x^3 + \sin t$$

Second order Runge-Kutta method – output



$N = 10, 20, 50, 100$ Convergence at $N = 50$ vs 1000 for Euler

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- Fourth order Runge-Kutta offers a balance between accuracy and ease to program and is considered to be the sweet spot.

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x(t+h) &= x(t) + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
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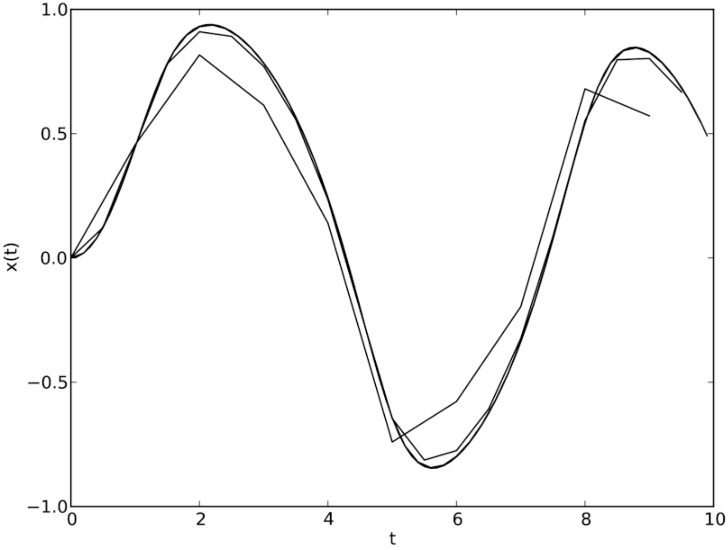

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Solution over infinite ranges

- In some cases, we want to march in time from some initial value to not some finite value in time, but to $t = \infty$.

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- With this substitution, as $u \to 1$, $t \to \infty$.

Solution over infinite ranges

$$\frac{dx}{dt} = f(x,t)$$

$$\text{Using Chain rule} \quad \frac{dx}{du}\frac{du}{dt} = f(x,t)$$

$$\frac{dx}{du} = \frac{dt}{du} f\left(x, \frac{u}{1-u}\right)$$

$$\text{But} \quad \frac{dt}{du} = \frac{1}{(1-u)^2}$$

$$\frac{dx}{du} = (1-u)^{-2} f\left(x, \frac{u}{1-u}\right)$$

$$\text{define} \quad g(x,u) \equiv (1-u)^{-2} f\left(x, \frac{u}{1-u}\right)$$

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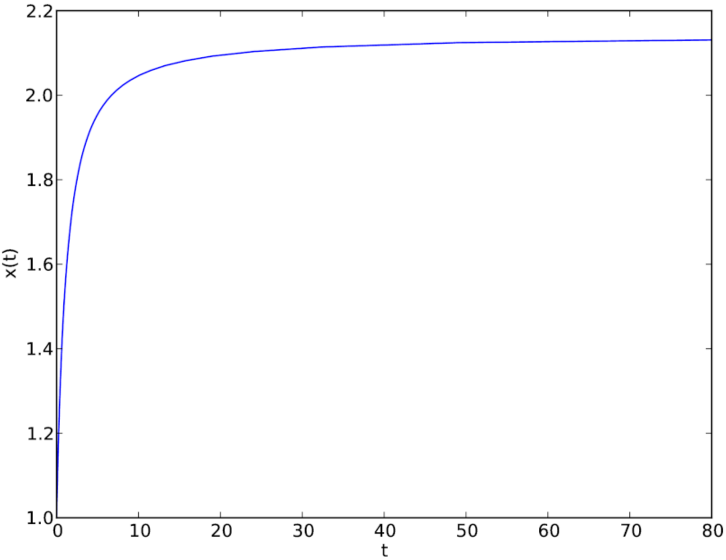
- Using the substitution:

$$\frac{dx}{du} = \frac{1}{x^2(1-u)^2 + u^2}$$

with $x = 1$ at $u = 0$ and range of u goes from $u = 0$ to $u = 1$.

Solution over infinite ranges – output



Differential equations with more than one variable

- In a lot of physics problems, we have more than one variable – ie we have simultaneous differential equations, where the derivative of each variable can depend on any or all of the variables as well as the independent variable, t:

$$\frac{dx}{dt} = f_x(x, y, t) \quad \frac{dy}{dt} = f_y(x, y, t)$$

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- These equations can be written in a vector form as:

$$\frac{d\mathbf{r}}{dt} = \mathbf{f}(\mathbf{r}, t)$$

  where $\mathbf{r} = (x, y, \ldots)$ and $\mathbf{f}$ is a vector of functions, $\mathbf{f}(\mathbf{r}, t) = (f_x(\mathbf{r}, t), f_y(\mathbf{r}, t), \ldots)$.

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- Fourth order Runge-Kutta:

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\end{aligned}$$

Simultaneous differential equations – example

Consider the following equations:

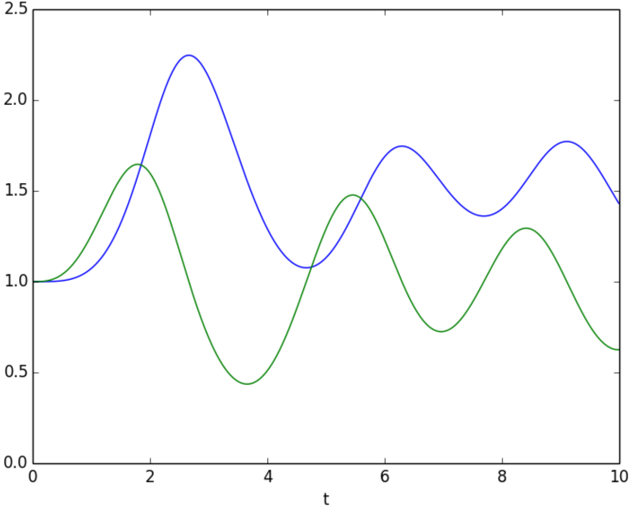
$$\frac{dx}{dt} = xy - x, \quad \frac{dy}{dt} = y - xy + \sin^2 \omega t$$

with initial conditions:

$$x = y = 1 \quad at \quad t = 0$$

and $\omega = 1$.

Simultaneous differential equations – code



Higher order differential equations

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- Similarly for 3rd order equation:

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- We can solve using methods we already know about simultaneous equations.