Random processes and Monte Carlo Simulation

- Monte Carlo integration.
- Non-uniform distributions.
- Random walk.

Mean value method

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- A simple way to estimate $\langle f \rangle$ is to just measure $f(x)$ at N points, $x_1, x_2, \ldots, x_N$ chosen uniformly between a and b:

$$\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

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- Variance on the sum is N times the variance on a single term or $N \mathrm{var} f$.
- Error/standard deviation on the integral:

$$\sigma = \frac{b-a}{N} \sqrt{N \mathrm{var} f} = (b-a) \frac{\sqrt{\mathrm{var} f}}{\sqrt{N}}$$

which goes as $1/\sqrt{N}$ but the variance is smaller!

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- Experiments with different types of distributions

How does one generate non-uniform random number distributions with a uniform random number generators?

Generating a non-uniform distribution with a probability distribution $w(x)$:

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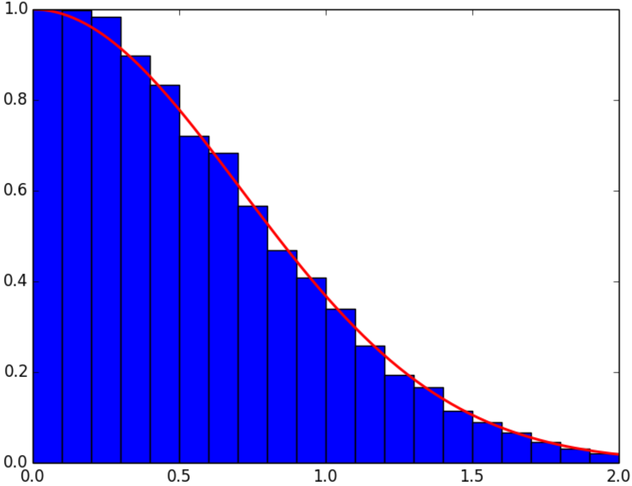
- The x_i so accepted will have the weighting $w(x_i)$.

von Neumann Rejection

$$w(x) = e^{-x^2} \quad x \in [0, 2]$$



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- Our goal is to choose the function $x(z)$ such that x has the distribution we want.

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- If we can do the integral on the left and solve the equation, we will have the required $x(z)$!

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Thus if we feed the above equation uniformly distributed z in interval $[0, 1]$, it will generate the exponential distribution x for us.

Gaussian random numbers

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- However, consider two independent random numbers x and y drawn from a Gaussian distribution with the same σ. The probability that a point (x, y) falls in an element $dxdy$ of the xy plane:

$$p(x)dx \times p(y)dy = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dxdy$$

Gaussian random numbers

- In polar coordinates:

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- Then one can construct x and y as:

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

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  - In three dimensions there are six possible neighbors.

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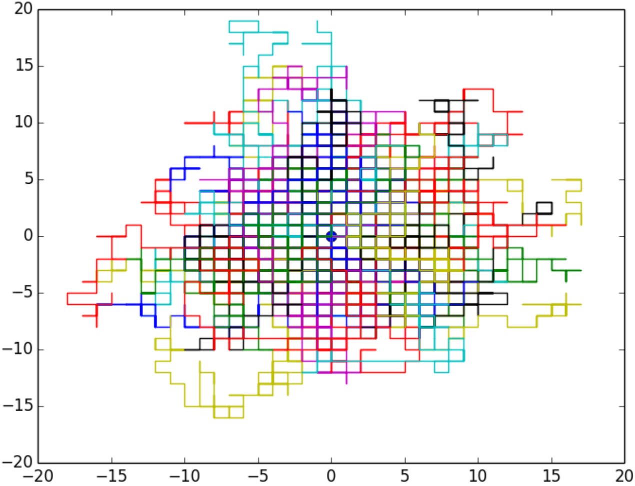
$$\text{In 100 steps}, \langle r \rangle \sim 8.9$$



Various models of random walk

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Examples of applications:
Spread of inflectional diseases and effects of immunization
Spreading of fire

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- Condition: the walker can not step back
- Goal: find average number of steps to get out the city

$$\text{To escape } 24 \times 24, \langle n \rangle \sim 92$$